# BULLETIN

OF THE

# AMERICAN MATHEMATICAL SOCIETY

A Historical and Critical Review
of Mathematical Science

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VOLUME XXXIV, NUMBER 4

JULY-AUGUST, 1928

PUBLISHED BY THE SOCIETY
MENASHA, WIS., AND NEW YORK
1928

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PUBLISHED BIMONTHLY

Whole No. 355 $7.00 a Year

Entered as second class matter at the post office at Menasha, Wis., May 13, 1926.

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1. Typewrite words and the very simplest formulas only.
2. *Do not* try to typewrite any complex formulas. Write them.
3. Keep a copy, and send the editors two copies, if you can.
4. *Do not* underline any symbols or any formulas.
5. Underline theorems with blue pencil (avoid ink).
6. Follow our recent styles in abbreviations, footnotes, etc.
7. Write carefully the (often misunderstood) capitals C K P S V W X Z.
8. Write $\epsilon$, not $\varepsilon$. Write very carefully $\gamma\ \eta\ \kappa\ \lambda\ \nu\ \tau\ \upsilon\ \chi\ \omega$.
9. Among Greek capitals, use only $\Gamma\ \Delta\ \Theta\ \Lambda\ \Xi\ \Pi\ \Sigma\ \Phi\ \Psi\ \Omega$.
10. Punctuate carefully, especially in formulas; thus: $1, 2, \cdots, n$.
11. Use the solidus (/) to avoid fractions in solid lines.
12. Use fractional exponents to avoid root signs everywhere.
13. Use extra symbols to avoid complicated exponents.
14. In typewritten formulas, $1$ means "one"; to indicate "ell" in formulas, backspace and overprint /; thus: $l$. Similarly, $0$ means "zero"; to indicate "cap O," backspace and overprint period; thus: $O$.
15. Avoid a dash over a letter, except for those shown below.
16. Some samples of unusual types available on monotype machines follow. A more complete list of all such types will be sent on request.

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Light Face Greek—$\alpha\ \beta\ \gamma \cdots$ (all) A B $\Gamma \cdots$ (all).

★ Light Greek Superiors—$^{\Delta}$ and $^{\alpha\ \beta\ \gamma \cdots}$ (all except $^{\iota}$ and $^{o}$).

★ Light Greek Inferiors—$_{\Delta\Lambda Z\Omega}$ and $_{\alpha\ \beta\ \gamma \cdots}$ (all except $_{\iota}$ and $_{o}$).

* Boldface Greek—$\boldsymbol{\alpha\ \beta\ \delta\ \varepsilon\ \zeta\ \eta\ \theta\ \mu\ \nu\ \xi\ \pi\ \varrho\ \sigma\ \omega}$ and $\boldsymbol{\Omega}$.

* Lightface German—$\mathfrak{a}\ \mathfrak{b}\ \mathfrak{c}\ \mathfrak{d}\ \mathfrak{p}\ \mathfrak{q}\ \mathfrak{A}\ \mathfrak{B}\ \mathfrak{C}\ \mathfrak{D}\ \mathfrak{E}\ \mathfrak{F}\ \mathfrak{G}\ \mathfrak{H}\ \mathfrak{J}\ \mathfrak{L}\ \mathfrak{M}\ \mathfrak{P}\ \mathfrak{Q}\ \mathfrak{R}\ \mathfrak{S}\ \mathfrak{T}\ \mathfrak{U}\ \mathfrak{V}\ \mathfrak{W}\ \mathfrak{X}\ \mathfrak{Y}\ \mathfrak{Z}$.

* Boldface German—$\mathfrak{b}\ \mathfrak{A}\ \mathfrak{B}\ \mathfrak{D}$.

Script (special font) $\mathcal{A}\ \mathcal{B}\ \mathcal{C} \cdots$ (all). No lower case manufactured.

* Hebrew— ℵ א ב troublesome to handle.

★ Dashed Italics—$\overline{A}\ \overline{a}\ \overline{B}\ \overline{b}\ \overline{C}\ \overline{c}\ \overline{E}\ \overline{e}\ \overline{F}\ \overline{f}\ \overline{G}\ \overline{g}\ \overline{H}\ \overline{I}\ \overline{i}\ \overline{K}\ \overline{k}\ \overline{M}\ \overline{m}\ \overline{N}\ \overline{n}\ \overline{P}\ \overline{p}\ \overline{q}\ \overline{R}\ \overline{r}\ \overline{s}\ \overline{t}\ \overline{u}\ \overline{v}\ \overline{X}\ \overline{x}\ \overline{Y}\ \overline{y}\ \overline{Z}\ \overline{z}$.

★ Tilda Italics—$\tilde{A}\ \tilde{a}\ \tilde{e}\ \tilde{N}\ \tilde{n}\ \tilde{O}\ \tilde{r}\ \tilde{u}\ \tilde{y}$.

★ Tilda Greek—$\tilde{\alpha}\ \tilde{\epsilon}\ \tilde{\eta}\ \tilde{\omega}\ \tilde{\varpi}$.

★ Dashed Greek—$\overline{\alpha}\ \overline{\beta}\ \overline{\gamma}\ \overline{\delta}\ \overline{\eta}\ \overline{\theta}\ \overline{\mu}\ \overline{\nu}\ \overline{\rho}\ \overline{\omega}\ \overline{\Gamma}$.

★ Dotted Italic—$\dot{a}\ \ddot{a}\ \dot{e}\ \ddot{e}\ \ddot{g}\ \ddot{\imath}\ \dot{m}\ \dot{n}\ \dot{q}\ \dot{r}\ \ddot{r}\ \dot{u}\ \ddot{u}\ \dot{x}\ \ddot{x}\ \dot{y}\ \ddot{y}\ \dot{z}\ \ddot{z}$.

★ Dotted Greek—$\dot{\eta}\ \ddot{\eta}\ \dot{\theta}\ \ddot{\theta}\ \dot{\xi}\ \dot{\psi}\ \ddot{\psi}\ \ddot{\omega}$ (single dotted $\zeta\ \phi\ \delta\ \beta\ \gamma$; double dotted $\gamma$ readily available).

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* Additional characters readily available at small cost.

★ Matrices for additional characters are made upon special orders and necessitate a delay of from four to eight weeks and average expense of $4.50 per matrix.

## THE APRIL MEETING IN CHICAGO

The two hundred sixty-first meeting, twenty-ninth regular Western meeting of the Society was held at the University of Chicago on Friday and Saturday, April 6–7, 1928. About one hundred twenty persons attended the meeting, among whom were the following ninety-six members of the Society:

A. A. Albert, N. L. Anderson, R. W. Babcock, Bamforth, Bardell, Barnett, Bartky, Bear, Bibb, G. A. Bliss, Bussey, G. N. Carmichael, R. D. Carmichael, Carver, Chaimovitch, Conkwright, Cook, Cope, H. B. Curtis, Curtiss, H. T. Davis, Davisson, Dickson, Douglas, Earl, Everett, A. L. Foster, C. A. Garabedian, H. L. Garabedian, Garver, Georges, Gladys Gibbens, J. W. Glover, Gouwens, L. M. Graves, Griffiths, V. G. Grove, Hartung, Hassler, Hyden, Louis Ingold, Ingraham, Dunham Jackson, B. W. Jones, Kinney, E. P. Lane, Langer, LaPaz, Laun, Logsdon, C. E. Love, Lunn, J. V. McKelvey, W. D. MacMillan, W. A. Manning, March, William Marshall, Marquis, T. E. Mason, Mickelson, C. N. Moore, E. H. Moore, E. J. Moulton, A. L. Nelson, Olson, Oppenheim, Palmié, Parkinson, Pettit, H. L. Rietz, Roever, Roth, Rupp, Sanger, Schottenfels, Sherer, Shohat, W. G. Simon, Skinner, Slaught, E. R. Smith, Virgil Snyder, Stouffer, J. H. Taylor, E. L. Thompson, Thornton, Trjitzinsky, J. S. Turner, Van Vleck, Wall, L. E. Ward, Warren Weaver, Wiggin, Roscoe Woods, Wyant, Yeaton.

On Friday afternoon Professors E. B. Stouffer and E. P. Lane gave symposium addresses on *Recent Developments in Projective Differential Geometry*. On a motion by Professor Carver, a rising vote of thanks was given the speakers. These addresses appear in full in the present issue of this Bulletin

On Friday evening members and their guests attended dinner in the Del Prado Hotel. Professor Jackson acted as toastmaster. At the dinner President Snyder announced the first award of the Frank Nelson Cole prize to Professor L. E. Dickson for his book *Algebren und ihre Zahlentheorie* and other work. President Snyder presented the prize to Professor Dickson, who thanked the President. Both speakers paid tributes to the memory of Professor Cole. The toastmaster also called upon Professor Slaught, Dr.

Griffiths and Professors Manning and C. N. Moore for talks. The dinner was attended by eighty-five persons.

The Council of the Society met Friday afternoon.

The following sixteen persons were elected to ordinary membership:

Mr. Charles Trotter Bunnell, University of Rochester;
Professor Andrew G. Clark, Colorado Agricultural College;
Professor Howard Adams DoBell, New York State College for Teachers;
Mr. Harold L. Dorwart, Yale University;
Miss Allegra Eckles, Territorial Normal School, Honolulu;
Mr. Edwin Harold Hadlock, Cornell University;
Professor Bruce Vickroy Hill, Phillips University;
Sister Mary Columba, Villa Maria College;
Sister Mary St. Bernard, Villa Maria College;
Mr. James Lewis Noecker, Thiel College;
Mr. Oscar John Peterson, University of Michigan;
Mr. Harry Ward Seward, Utica, N. Y.;
Professor Clarence DeWitt Smith, Louisiana College;
Professor Elva Elizabeth Starr, Alfred University;
Mr. John Russell Vatnsdal, State College of Washington;
Mr. Clement Weinstein, University of Pennsylvania.

The following were elected to membership as nominees of Members of Department of Mathematics, University of Wisconsin:

Mr. C. W. Dancer, University of Wisconsin;
Mr. G. N. Carmichael, University of Wisconsin;
Mr. R. O. Gilmore, University of Wisconsin.

The President announced as the Committee on Arrangements for the semi-centennial celebration of the Society: Professors T. S. Fiske (Chairman), R. C. Archibald, J. L. Coolidge, L. E. Dickson, E. R. Hedrick, Dunham Jackson, James Pierpont, M. I. Pupin, R. G. D. Richardson, and Oswald Veblen.

The following were appointed as a Committee for the nomination of officers to be elected at the next annual meeting: Professors L. E. Dickson (Chairman), E. T. Bell, D. R. Curtiss, H. M. Morse, J. K. Whittemore.

It was decided to hold a fall meeting at Cincinnati this year. It is proposed, if this meeting shows the need, to hold such a meeting each year in the Ohio Valley Region. This

meeting will probably take place on the Friday and Saturday after Thanksgiving. The President appointed as a Committee on Arrangements to act with the Associate Secretary: Professors C. N. Moore (Chairman), Henry Blumberg, P. P. Boyd, Louis Brand, and H. T. Davis.

The following dates for meetings of the Society in the Middle West were determined upon: December 31-January 1, 1928-9, University of Chicago; March 28-29, 1929, University of Chicago. As a Committee to arrange for a symposium at the latter of these meetings, the President appointed Professors E. P. Lane (Chairman), M. H. Ingraham, C. C. MacDuffee.

It was determined to hold the Western winter meeting of the Society for 1929-30 at Des Moines in connection with with the meeting of the American Association for the Advancement of Science.

The following were appointed as a Committee to study the question of how to increase the value of the summer meetings: Professors Arnold Dresden (Chairman), C. R. Adams, G. C. Evans, L. M. Graves, A. J. Kempner, and Warren Weaver.

The Council unanimously voted to thank the Carnegie Corporation for its recent gift of $5,000 to the endowment of the Society.

At the Saturday morning meeting of the Society the following resolution expressing the Society's appreciation of the action of the Mathematical Association of America in becoming a sustaining member was unanimously passed:

The American Mathematical Society wishes to express its particular appreciation of the constant and strong support that the Mathematical Association of America has given it, and especially of its gracious action in becoming a sustaining member of the Society It hopes that this unity may strengthen the efforts of the two organizations to further mathematics in America.

The papers whose abstracts appear below were read in three sections: (1) Geometry and Applied Mathematics—

Friday morning; (2) Algebra and Number Theory—Friday morning; (3) General, mostly Analysis—Saturday morning. Professor Roever presided at the first section, Professor C. N. Moore at the second, President Snyder and Professor Stouffer at the third. President Snyder presided at the symposium on Friday afternoon. Papers numbered 1–20 were read in the section on Geometry and Applied Mathematics, 21–30 in the section on Algebra and Number Theory, and 31–49 Saturday morning. Papers numbered 3, 8–10, 12, 14–19, 42–49 were read by title. Professor Hufford was introduced by Professor Davis, and Mr. Shook, Mr. Yank, and Mr. Morrow were introduced by Professor Dickson.

1. Professor I. A. Barnett: *The finite transformations generated by an infinitesimal projective transformation in function space.*

In a paper entitled *Projective transformations in function space*, Transactions of this Society, 1919, Professor L. L. Dines gave a method for determining the one-parameter family of projective transformations generated by a regular infinitesimal projective transformation. The method, however, made use of certain identities involving the inverse of a projective transformation in function space. In this paper, it is shown that by the introduction of homogeneous coordinates in the infinitesimal transformation, it is possible to avoid the formalities which Dines uses and obtain the result much more briefly and directly. For the particular case of the sub-group which leaves the unit sphere in function space invariant, the problem is reduced to integrating two systems of two simultaneous integro-differential equations.

2. Professor J. O. Hassler: *Plane nets whose first and minus first Laplacian transforms each degenerate into a straight line.*

This paper will appear in full in an early issue of this Bulletin.

3. Professor Arnold Emch: *Multiple systems, their geometric representation and groups.*

This paper considers triple and multiple systems with particular reference to their geometric representations and group properties. Two triple systems each invariant under a $G_{168}$ on 7 elements without a common triple may be mapped on a torus. The configuration of triangles thus obtained is connected with the 7 hexagons on the torus invariant under a $G_{42}$. There exist triple systems such that all possible triples may be arranged in a complete set of triple systems, in which every triple occurs just once. This fact as shown by the author in case of 9 elements seems to be new. Two triple systems on 9 elements without a common triple may

be mapped on a torus. The group leaving a triple system on 9 elements invariant is a $G_{432}$. The configuration on the torus is connected with a polyhedron on the torus bounded by 6 octagons and 6 squares and is invariant in a non-cyclic $G_6$. A very interesting configuration is obtained by the mapping of a triple system on 6 elements in which every couple occurs twice. We obtain a polyhedron on the projective plane bounded by 6 pentagons which is invariant in an icosahedral $G_{60}$. The lowest non-trivial (1234) quadruple system, containing every triple one, is on 8 elements. Such a quadruple system is invariant under a $G_{1344}$. Two such quadruple systems without a common triple may be mapped into a hyperpolyhedron in space of four dimensions, which is invariant in a $G_{2.6}$. The quadruple system on 16 elements containing every element 6 times contains 24 quadruples and is invariant under a $G_{192}$. Its map is the four-dimensional cube.

4. Professor Virgil Snyder: *Quadratic involutions in multiple linear complexes.*

It is shown that space involutorial birational transformations exist having the three properties: (a) Lines joining a pair $PP'$ of conjugate points belong to a non-singular linear complex; (b) each line of the complex contains $k$ such pairs of conjugate points, $k$ any positive integer; (c) such transformations exist which have no surface of invariant points. Every plane contains a self-conjugate curve of order $2k+1$, and the correspondence $PP'$ on it is always singular for $k>1$.

5. Dr. Jesse Douglas (National Research Fellow): *Reduction of the problem of Plateau to the minimization of a certain functional.*

In a previous paper (abstract in this Bulletin, March-April, 1927) the author shows how to reduce the Plateau problem to the integral equation $(1)^*\int_0^1 K(t,\tau)\,\mathrm{ctn}\,\frac{1}{2}(\phi(\tau)-\phi(t))d\tau=0$, or to the equivalent integral equation $^*\int_0^1 K(t,\tau)d\tau/(\psi(\tau)-\psi(t))=0$, where $\psi(t)=\mathrm{ctn}\,\frac{1}{2}\phi(t)$. The function $K(t,\tau)$ is known as soon as the contour is given. The star indicates that the improper integral is to be taken in its Cauchy principal value. The function $\phi(t)$ is required to be always increasing and continuous as t increases from 0 to 1, and $\phi(0)=0$, $\phi(1)=2\pi$. The present paper points out that (1) is the Euler-Lagrange equation for the minimization of the functional $A(\phi)=-\int_0^1\int_0^1 K(t,\tau)\,\log\sin\frac{1}{2}|\phi(\tau)-\phi(t)|\,dt\,d\tau$; that is, (1) expresses the vanishing of the Volterra derivative $A'(\phi,t)$. An existence theorem is established for this minimum problem in case $K(t,\tau)$ is positive for all $t,\tau$ on the basis of the ideas in Fréchet's thesis (Palermo Rendiconti, vol. 22 (1906)), relative to functions on closed, compact sets.

6. Dr. Jesse Douglas: *A characterization of Riemann spaces of constant non-zero curvature and $n\geqq 3$ dimensions.*

Consider a projective space $P_n$ of $n\geqq 3$ dimensions, referred to coordinates $x^i (i=1, 2, \cdots, n)$. A metric for angles may be imposed on $P_n$ by means of a relative quadratic form $A=\nu^2 g_{ij}dx^i dx^j$, where the $g_{ij}$ are

fixed functions and $\nu^2$ is an arbitrarily variable function of the $x$'s; hence only the ratios of the $g$'s are important. A metric for areas may be imposed on $P_n$ by defining the area of the infinitesimal parallelogram determined by the elements $dx$, $\delta x$, to be the square root of $F=A_{ij,kl}(dx^i\delta x^j - dx^j\delta x^i)(dx^k\delta x^l - dx^l\delta x^k)$, where $A_{ij,kl}$ are arbitrarily assigned functions of the $x$'s. An angle-area space is identifiable with a Riemann space when and only when the $A_{ij,kl}$ are proportional to the two-rowed determinants $g_{ik}g_{jl}-g_{il}g_{jk}$. The present paper shows that if $n\geqq 3$, the only way to impose upon $P_n$ an angle-area metric which shall be such that in every rectilinear triangle the angular excess is equal to a non-zero constant times the area, is to define $A$, $F$ as in the Cayley metric based on any fixed quadric in $P_n$.

7. Dr. Jesse Douglas: *A new special form for the linear element of a surface.*

This paper deals with the case $n=2$ of the problem of the preceding paper. If $n=2$, any $A$, $F$ metric is identifiable with a Riemann metric by a proper choice of the factor $\nu^2$. Thus we have to find what the linear element of a surface must be in order that there may exist upon it a system of paths having the following two properties: (1) the system shall be linear (equivalent by point transformation to the straight lines of a plane); (2) the angular excess of any triangle of paths shall be proportional to its area, with a constant of proportionality $k\neq 0$. This problem is solved, a formula being obtained for $ds^2$ which involves four arbitrary functions: $U_1$, $U_2$ of $u$ alone, and $V_1$, $V_2$ of $v$ alone. Only one of these functions is essential, if we assume the surface to be real. Many surfaces not of constant Gaussian curvature are included.

8. Dr. Jesse Douglas: *Necessary and sufficient conditions for the equivalence of general affine and descriptive spaces of paths.*

This paper is a generalization of the work of Christoffel on the equivalence of quadratic differential forms, and of Veblen and J. M. Thomas on the equivalence of "restricted" spaces of paths, that is, systems of paths definable by $d^2x^i/dt^2=\Gamma_{jk}^i(dx^j/dt)(dx^k/dt)$, where the $\Gamma$'s are any functions of the $x$'s. For references, see Chapter V of O. Veblen, *Invariants of Quadratic Differential Forms*, Cambridge, 1927. General affine and descriptive spaces of paths have been defined by the writer in his paper *The general geometry of paths*, soon to appear in the Annals of Mathematics. The analytic basis here is a system of differential equations of the form $d^2x^i/dt^2=H_2^i(x, dx/dt)$, where $H_2^i$ denotes any function (generally transcendental) homogeneous of the second degree in the arguments $dx/dt$. The present paper develops criteria for the affine and for the descriptive equivalence of two such general spaces of paths.

9. Dr. Jesse Douglas: *Normal coordinates for a space of $K$-spreads.*

Normal coordinates for the "restricted" geometry of paths: $d^2x^i/dt^2 = \Gamma_{jk}^i(x)(dx^j/dt)(dx^k/dt)$, generalizing Riemann's normal coordinates, have been defined by O. Veblen, (Proceedings of the National Academy, vol. 8 (1922)). Normal coordinates for the general geometry of paths: $d^2x^i/dt^2 = H_2^i(x, dx/dt)$, have been defined by the writer in his paper cited in the preceding abstract. In the present paper, normal coordinates are defined for a space of $K$-spreads (abstract, this Bulletin, Jan.-Feb. 1928). They are relative to a coordinate system $(x)$, a fixed point $x_0$, and a fixed $K$-element at $x_0$. It is proved that these normal coordinates have the essential property that when the coordinate system $(x)$ undergoes arbitrary analytic transformation, the related normal coordinates $y$ undergo linear homogeneous transformation. This property enables us to define the process of *extension* of a tensor by means of partial differentiation with respect to the normal coordinates.

10. Dr. Jesse Douglas: *Differentiation with respect to a parameter of the Cauchy principal value of an improper integral.*

It is found that formal differentiation under the integral sign does not apply to the Cauchy principal value of an improper integral, even when the functions involved are quite regular. Thus: $(d/dt)^*(\int_0^1 K(t, \tau)(\tau - t)^{-1}d\tau = {}^*\int_0^1 (\partial K/\partial t + \partial K/\partial \tau)(\tau - t)^{-1}d\tau$. This result is generalized to the case ${}^*\int_0^1 K(t, \tau)F(\phi(\tau) - \phi(t))d\tau$, where $F$ has an infinity of order one at the zero value of its argument.

11. Professors H. T. Davis and M. E. Hufford: *Diffraction and the wave theory of light.*

The purpose of this paper is to check the energy in a diffraction pattern of 66 rings secured by passing monochromatic light through a circular orifice. The assumptions made are those of the classical wave theory of light and the mathematical development follows that given in the celebrated papers of E. Lommel. (See Gray, Mathews and MacRobert, *Bessel Functions*, 1922, Chapter 14.) The mathematical interest centers in the discovery of divergent series asymptotic to the Lommel functions $U_1(x)$ and $U_2(x)$. The striking agreement of the calculated energy with that found in the diffraction pattern, particularly as shown in the broadening and darkening of the rings toward the outer part of the plate, gives renewed confidence in the classical theory of light as applied to diffraction phenomena.

12. Dr. C. D. Smith: *On generalized Tchebycheff inequalities in mathematical statistics.*

It is the main purpose of the paper to give a further development of the theory and properties of what may be appropriately called generalized Tchebycheff inequalities. The inequality is discussed as a proposition of geometry. The Markoff Lemma is generalized in such a manner as to give rise to several important forms of the inequality in a very simple way. Some closer inequalities are derived. It is shown that the closeness of the

inequality may be improved in certain cases by moving the origin. Certain types of distributions are discussed to show that rather simple types of distributions exist within which the difference between the members of the inequality attains a minimum. Types of functions are discussed for which the difference between the members of the inequality may be made arbitrarily small.

13. Professor H. W. March: *The torsion problem for prisms of non-isotropic material.*

It is shown by means of a linear transformation that the soap film method can be used to solve the torsion problem for a prism composed of non-isotropic material having three mutually perpendicular planes of elastic symmetry, two of which are parallel to the length of the prism. The torsional rigidity of the given prism is equal to that of a transformed prism of isotropic material whose modulus of rigidity is expressed in terms of the moduli of the material of the non-isotropic prism. The lines of shearing stress in the cross section of the transformed prism become, after transformation to the cross section of the original prism, the lines of shearing stress in that section.

14. Professor E. L. Dodd: *A test for periods when certain measurements are of two variates combined.*

The usual tests in periodogram analysis are decidedly untrustworthy in dealing with data where some measurements probably represent two variates erroneously united. Such an error may occur in measuring the thickness of layers not always distinctly separated. For such data, an "intensity" should be defined (1) not greatly affected by the above slipping of phase, (2) not too difficult to compute, (3) with expected value zero, and (4) with mean error unity for every trial period. The following definitions for the intensity $I$ meets requirements (1), (3), and (4), and perhaps (2). Let the data be divided into sets of $s$ variates, for example, $s=60$ to test for a period of $k$. Suppose $m=s/k$ is an integer. Take $t=2\pi/k$. Then for each of the $m$ sub-sets of $k$ variates $X_r$, compute, with $r=0, 1, \cdots, k-1$, $C=\sum X_r \cos rt$, $S=\sum X_r \sin rt$, $U=C^2+S^2$. Let $a$ be the second moment of the $s$ variates about the mean, $b$ the fourth moment. With $k>2$, take $A^2=kb+(k^2-3k)a$, $I=(\sum U-sa)/(Am^{1/2})$. Suitable modifications are possible for $k=2$, and for fractional $m$; and the $I$'s may be combined for successive sets of $s$ variates.

15. Professor C. C. Camp: *Note on unexpected errors in Barlow's tables and others.*

One should not expect to find an error in a list of errata. M. J. Perrott in writing of Steinhauser's twenty-place table says "log 1088 doit terminer par 26 et non par 23." Actually, 226 occurs after the nineteenth decimal and Perrott must have miscounted. In the 30th edition of Schrön's table an inversion of the correct digits 42 occurs in his extended form of log 102238. Barlow's Tables as revised in 1839 were supposed to have errors of unity in the last digit of square and cube roots of some numbers beyond 1250. For

numbers under 1250 De Morgan says that "an error of a unit in the last place can . . . . hardly exist . . . . , though it is barely possible in numbers not much below 1250." An error is found in the cube root of 80, also it was found that the cube roots of 12, 20 were too large and those of 9, 14, 52 too small by unity in the last digit as given, the actual errors being greater than that in three cases. Moreover, errors exist in nine related cube roots. Such tables, which are widely copied, should be made entirely free of error.

16. Mr. J. H. Roberts: *Certain types of atriodic continua.*

R. L. Moore has introduced the notion *triodic continuum* and shown that if $G$ is an uncountable set of triodic continua in the plane, then $G$ contains an uncountable subset $G_1$, such that every two continua of $G_1$ have a point in common. A continuum $M$ is said to be triodic if it contains four distinct continua $a$, $b$, $c$ and $k$, such that $k$ is the common part of each two of the continua $a$, $b$ and $c$. Professor Moore has raised the following question: Is it true that if $M$ is a plane atriodic continuum, then there exists in the plane an uncountable set $G$ of mutually exclusive continua such that each continuum of $G$ can be thrown into $M$ by a continuous one-to-one transformation of the plane into itself. This is shown to be true for the special case where, given any positive number $\epsilon$, $M$ can be covered by a simple chain of regions of diameter less than $\epsilon$. An example is given of an *unbounded* atriodic continuum $M$ for which no such set $G$ exists.

17. Professor G. T. Whyburn: *A theorem on plane continua.*

In this paper the following theorem is proved. Suppose $M$ is a plane continuum having the property that each of its subcontinua contains a continuum $N$ such that there exists a positive number $d$ such that every subcontinuum of $N$ contains at least one point which is accessible from at least two complementary domains of $M$ which are of diameter $>d$; then $M$ is a continuous curve. A number of interesting corollaries of this theorem are given, among which are the following: (1) Under the same hypothesis as above, (a) every subcontinuum of $M$ contains an arc which belongs to the common boundary of two of the complementary domains of $M$, and (b) $M$ is a Menger regular curve. (2) If every subcontinuum of $M$ contains a point which is accessible from two unbounded complementary domains of $M$, then every point of $M$ is a cut point of $M$.

18. Professor G. T. Whyburn: *Some theorems on connected point sets.*

In this paper the following generalizations of theorems due to R. L. Wilder (Fundamenta Mathematicae, vol. 7, p. 371), R. L. Moore (Proceedings of the National Academy, vol. 9, p. 102, Theorem B+), and Kuratowski and Zarankiewicz (this Bulletin, vol. 33, p. 574, Remark B) are given. (1) Condition (4) in the Moore-Wilder lemma characterizing continua which are not continuous curves and proved by Wilder only for bounded continua is shown to hold for unbounded continua. (2) No connected subset $K$ of a connected point set $M$ contains an uncountable

collection of mutually exclusive point sets each of which cuts $M$ but not $K$. (3) If $S$ is any connected point set and $Z$ is any collection of mutually exclusive connected subsets of $S$ such that for each element $X$ of $Z$, $M-X$ is neither connected nor the sum of two connected point sets, then $Z$ is countable. (Kuratowski and Zarankiewicz imposed the unnecessary condition that each set of the collection $Z$ is closed relative to $S$.) In proving (1) use was made of the following lemma: If $N$ is a connected subset of a connected point set $M$ and $K$ is any component of $M-N$, then $M-K$ is connected.

19. Professor G. T. Whyburn: *Concerning collections of the cuttings of continua.*

In this paper the following results are established. Let $M$ denote any plane continuum. (1) No plane continuum $M$ contains an uncountable collection of mutually exclusive connected point sets each of which contains a proper subset which cuts $M$; hence if $G$ is any collection of mutually exclusive connected subsets of $M$ each of which contains a cutting of $M$, then all save possibly a countable number of the elements of $G$ must be continua and must themselves be irreducible cuttings of $M$. (2) If $M$ is bounded and $G$ is any collection of mutually exclusive componentwise irreducible cuttings of $M$ such that for each element $g$ of $G$, $M-g$ is not the sum of two connected point sets, then $G$ is countable. (3) If $G$ is any collection of mutually exclusive bounded subcontinua of $M$ each of which contains a cutting of $M$, then $G$ contains a subcollection $G^*$ which contains all but a countable number of the elements of $G$ and which is an upper semi-continuous collection.

20. Professor C. A. Rupp: *On an extension of Pascal's theorem to a space of n dimensions.*

The edges of a simplex $F$ meet a hyperquadric $V$ in $S_r$ in $r(r+1)$ points, which may be grouped to define a second simplex $P$. The coordinates of the intersections of corresponding faces of the simplexes are linearly dependent. This is shown to be equivalent to saying that the pair of simplexes $F$ and $P$ are a Schlafli pair, or that lines in the position of Schlafli are linearly dependent. Applying the theorem to circles gives a new construction for a Stephanos pentacycle, and shows that the circles of a pentacycle are linearly dependent. In line geometry there appears the concept of linear congruences in involution, and by means of six linear complexes in involution are defined six linear congruences such that any linear congruence in involution with five of them is also in involution with the sixth.

21. Mr. A. A. Albert: *A determination of all normal division algebras in sixteen units.*

It is known that every normal division algebra in 16 units is of rank 4 and contains an element satisfying an irreducible quartic in the reference field $F$. In this paper the quartic is taken in the reduced form and it is shown that either all of its roots are rational functions of one of them or the algebra contains an element satisfying the equation $\phi(\omega)=\omega^4+\alpha\omega^2$

$+\beta$, $\alpha$, $\beta$ in $F$, which is irreducible in $F$. It is then shown that either this equation has the property that all of its roots are rational functions of one of them or its group is $G_8$. The algebras containing an element satisfying an equation of this type with group $G_8$ are then determined, and hence all division algebras in sixteen units, since those algebras which contain an element satisfying an irreducible equation having the property that all of its roots are rational functions of one of them are the algebras of type $\Gamma$ of L. E. Dickson and are known.

22. Professor W. A. Manning: *A theorem on simply transitive primitive groups.*

Let $G_1$ be a subgroup that fixes one letter of a simply transitive primitive permutation-group. Beginning in 1917, the author published a series of theorems concerning the structure of $G_1$ when one of its transitive constituents is doubly transitive. The most recently published theorem (1921) states that if $G_1$ has a doubly transitive constituent of degree $m$, either $G_1$ is a simple isomorphism between doubly transitive groups of degree $m$, or $G_1$ has a transitive constituent whose degree is a divisor $(>m)$ of $m(m-1)$. His present theorem asserts that if $G_1$ has a doubly transitive constituent of degree $m$, $G_1$ has also a transitive constituent whose degree is a divisor $(>m)$ of $m(m-1)$.

23. Professor L. E. Dickson: *The general Waring problem for quadratic functions.*

This paper reports on an elaborate investigation of the following difficult problem. Let the quadratic function $q(x)$ have a positive coefficient of $x^2$, be an integer for every integer $x \geqq 0$, and be negative for one or more integers $x \geqq 0$. There is found the least integer $E$ such that every integer $\geqq L$ is a sum of $E$ numbers chosen from 0 to 1 and four values $\geqq 0$ of $q(x)$ for integers $x \geqq 0$, where $L$ denotes the least sum of four such values. In previous papers, the writer treated the case in which $q(x)$ is never negative when $x$ is positive.

24. Mr. R. C. Shook: *Two extended Waring problems.*

The first problem considers the conditions that a form $f=a_1x_1^6+\cdots+a_nx_n^6$, $x_i \geqq 0$, shall satisfy in order that it may represent all integers. A minimum order $n=10$ is established and twenty-two forms of order 10 are shown to represent all integers up to 4,096. The second problem discusses the maximum gaps in a table of sums by four of (i) values of quadratic functions $f(x)$, having integral values $\geqq 0$ for $x \geqq 0$, and (ii) the positive values of quadratic functions $q(x)$ having integral values not all equal to or greater than zero for $x \geqq 0$.

25. Professor L. E. Dickson: *Positive binary quadratic forms $B$ such that every positive integer is a sum of $s$ values of $B$.*

Since $B$ must represent unity, we may transform $B$ into $f=x^2+gxy+hy^2$ where $g=0$ or 1, and $h>0$. Hence the case $s>3$ is trivial. For $s=2$ or 3, it

is shown that every positive integer is a sum of $s$ values of $f$ if and only if $h=1$, 2, or 3 if $s=2$, and $h=1, \cdots, 7$ if $s=3$. The paper will appear in Journal de Mathématiques.

26. Professor L. E. Dickson: *Generalization of Waring's theorem on cubes.*

This paper contains a further consideration of the theorems announced in the American Mathematical Monthy, April, 1927.

27. Mr. K. C. Yang: *All positive integers are sums of 9 pyramidal numbers* $(x^3-x)/6$, $x=0, 1, 2, \cdots$.

E. Maillet, in the Bulletin de la Société de France, vol. 23, (1895), pp. 40-49, proved that every integer $\geq 19272$ is a sum of at most 12 pyramidal numbers. By an elementary proof, it is here shown that nine suffice.

28. Dr. Lois W. Griffiths: *Representations of integers in the form* $x^2+2y^2+3z^2+6w^2$.

The number of representations $2T(N)$ of a positive odd integer $N$ in the form $x^2+2y^2+3z^2+6w^2$ is not known exactly. In this paper new conditions on, and new limits for, $T(N)$ are obtained in terms of the divisors of $N$. The determination of $T(N)$ for $N$ odd and composite is reduced to that of $T(P)$, where $P$ is in turn the distinct prime factors of $N$. The paper is to appear in the American Journal of Mathematics.

29. Mr. B. W. Jones: *Representation of integers by positive ternary quadratic forms.*

L. E. Dickson (Annals of Mathematics, (2), vol. 28 (1927), p. 333) made the following definition: "All the integers not represented by a *regular form* $f[=ax^2+by^2+cz^2$, where $a$, $b$ and $c$ are positive integers] coincide with all the positive integers given by certain arithmetical progressions," and proved that not more than seventeen forms $f$ are regular, where $a=1$, $b$ and $c$ relatively prime and less than certain large integers. Applying his methods and extension of them, this paper proves first that not more than 103 forms $f$ are regular when 1 is the greatest common divisor of $a$, $b$ and $c$. Then, using methods of Dirichlet and Dickson, and certain extensions of these, together with elementary transformations and previously known results for certain ternaries, all but six of these forms are proved regular. (Partial results are known for these six.) Many quadratic ternary forms with cross products are similarly proved regular. Also numerous semi-regular forms are considered.

30. Mr. D. C. Morrow: *All quaternary quadratic forms which represent every positive integer.*

It is proved that all positive quaternary quadratic forms are equivalent to certain reduced forms. For forms which represent all positive integers, it is shown that the number of reduced forms is finite. The conditions on the coefficients, which are necessary in order that the re-

duced form shall represent every positive integer, are derived, and certain equivalent cases are eliminated. The paper then proceeds to determine all those reduced forms which represent every positive integer.

31. Dr. F. R. Bamforth: *A classification of boundary value problems for a system of ordinary differential equations of the second order.*

Boundary value problems of the form $y'(x)=(A(x)+\lambda B(x))y(x)$, $My(a)+Ny(b)=0$, are considered where $\lambda$ is a parameter, capital letters denote square matrices of order two, and small letters vectors. It is shown that every boundary value problem of this form, where the matrix $B(x)$ satisfies certain hypotheses, can be reduced to one and only one of a set of four normal forms. This reduction can be made by means of a non-singular transformation of the form $y=U_n$. Necessary and sufficient conditions are found for a boundary value problem in one of these normal forms to be self-adjoint according to the definition of G. A. Bliss. It is pointed out which of the normal forms have been studied and a boundary value problem in one of these forms, hitherto not studied, is solved completely. Finally, a general boundary value problem is deduced from an isoperimetric problem in the calculus of variations.

32. Dr. L. E. Ward: *On third order boundary value and expansion problems.*

A classification of boundary value problems of the system consisting of the differential equation $u'''+\rho^3 u=0$ and three boundary conditions linear and homogeneous in the values of $u$ and its first two derivatives at two real points is given. The forms of the boundary conditions in one of the irregular cases are considered, and some of the properties of an infinite series of the characteristic functions arising from this case are derived.

33. Dr. L. E. Ward: *On the uniqueness of the coefficients in a certain expansion problem.*

In a paper published in the Transactions of this Society, vol. 29 (1927), the author showed that a certain irregular boundary value problem leads to characteristic functions in an infinite series in which any analytic function can be expanded, and that the formal series for the function converges uniformly to the function in the interior of a certain triangle. In this paper it is shown that zero can be expanded in such a series, uniformly convergent in the interior of the same triangle, and that therefore the coefficients in this type of expansion are not unique.

34. Mr. T. F. Cope: *An analogue of the Jacobi condition for the problem of Mayer with variable end points.*

In this paper the generalized Mayer problem formulated by G. A. Bliss (Transactions of this Society, vol. 19 (1918), p. 305) is considered. The second variation for this problem is computed and reduced to a particularly simple form. It is then shown that there is a boundary value

problem associated with the second variation, from which a necessary condition for a solution of the original problem is deduced. This condition is, essentially, that for a minimizing arc for the original problem, the boundary value problem can have no solution for negative values of its parameter. Finally, the boundary value problem associated with the second variation is transformed into an equivalent problem, which is shown to be definitely self-adjoint according to the definition of G. A. Bliss (Transactions of this Society, vol. 28 (1926), p. 570).

35. Dr. W. J. Trjitzinsky: *A class of quasi-analytic functions.*

In the field of quasi-analytic functions (they are non-analytic, but determined by their initial values at a point) some of the unsolved questions are: (1) construction of such functions, (2) determination of conditions under which an assigned set of initial values defines such a function. In this paper it is shown that functions of the form $\sum a_n/(x-c_n)$ are quasi-analytic in the real interval $(-1, +1)$, if $\sum_n |a_n| \log_k^{n+1} n$ $(k \geqq 2)$ converges and $c_n$ is an enumerable set of points whose limiting points are everywhere dense on $(-1, +1)$ and whose distances from the real axis approach zero not faster than $1/\log_k n$. Using this result, sufficient conditions are found for the existence of quasi-analytic functions when a set of initial values is assigned.

36. Professor C. N. Moore: *On Gibbs' phenomenon for the developments in Bessel's functions.*

In a paper of 1911 (Transactions of this Society, vol. 12, pp. 181-206) the author developed an asymptotic formula for the coefficients of the developments in Bessel's functions which made it possible to infer facts concerning the convergence of these developments in the neighborhood of the origin which have not as yet been obtained in any other manner. In the present paper it is shown that the nature of the Gibbs' phenomenon in the neighborhood of the origin for these developments can be readily deduced from the same asymptotic formula.

37. Professor Dunham Jackson: *A note on closest approximation.*

This paper is concerned with a variation of the problem of closest approximation, which may be formulated as follows: Let $f(x)$ be a given continuous function of period $2\pi$. Let $h$ be a given positive number—less than or greater than $2\pi$. (The value $h=2\pi$ is not excluded of necessity, but because the results in this case are already known.) For a given trigonometric sum $t_n(x)$, of the $n$th order, let $I(t_n)$ be the maximum of the integral of $[f(x)-t_n(x)]^2$ over an interval of length $h$, as the initial point of the interval is allowed to vary. Among all trigonometric sums of the $n$th order, let $T_n(x)$ be the one for which $I$ has the smallest possible value. The minimum problem has a unique solution, and the minimizing sum $T_n(x)$ converges uniformly toward $f(x)$ as $n$ becomes infinite, if $f(x)$ satisfies suitable hypotheses (for example, if $f(x)$ satisfies a Lipschitz

condition). The problem can be greatly generalized without increasing the difficulty of the treatment.

38. Professor J. A. Shohat: *On the convergence of mechanical quadratures on an infinite interval.*

A function $\Psi(x)$ defined on an interval $(a, b)$, finite or infinite non-decreasing, with infinitely many points of increase, gives rise to a system of orthogonal Tchebycheff polynomials $\{\phi_n(x)\}$ of degree $n\ (=0, 1, 2, \cdots)$ having each roots $(x_i)$ real, distinct and between $a, b$. We then construct the formula of "Mechanical Quadratures"

$$\int_a^b f(x)d\Psi(x) = \sum_{i=1}^{n} H_i f(x_i) + R_n(f), \tag{1}$$

where $R_n(f)\equiv 0$, if $f(x)$ is a polynomial of degree $<2n$. In the present paper the author investigates the convergence of formula (1), that is, for what classes of functions $f(x)$ is $\lim_{n\to\infty} R_n(f)=0$. A complete solution is given in the most important case: $\Psi(x)$ is continuous in $(a, b)$, which includes that of $\int_a^b f(x)p(x)dx$, as a particular case. The case $(a, b)$ finite is herein included, thus generalizing a result due to Stieltjes. Applications are given to Tchebycheff polynomials, also to the theory of probability. Part of these results appeared in a note in the Comptes Rendus, Feb., 1928.

39. Professor J. A. Shohat: *On the convergence of the expansion of continuous functions in series of Sturm-Liouville functions.*

Consider the expansion $f(x) = \sum_{n=1}^{\infty} A_n U_n(x)$, $(A_n = \int_a^b f(x)\, U_n(x)dx)$, where $U_n(x)$ denotes the solution of the reduced Sturm-Liouville differential equation with the known boundary conditions. The author shows that the the convergence of the expansion (1) can be settled immediately, by means of the most elementary propositions of integral calculus, in case $f(x)$ satisfies a Lipschitz condition. In many cases we thus readily derive the order (with respect to $n$) of $A_n$ and of the remainder in (1). The method is applicable to differential equations of higher order, also to many other expansions in series of orthogonal functions.

40. Mr. W. C. Risselman: *On the approximate representation of a given function by means of polynomials in another given function.*

The problem is that of approximating to a given function $f(x)$ by means of a polynomial $P_n$ of given degree in a function $\phi(x)$ so as to mimimize $\int |f(x)-P_n[\phi(x)]|^m dx$, the integral being taken between finite limits $(a, b)$. On the assumption that the inverse of $\phi(x)$ is single-valued and under certain further hypotheses, which are less restrictive in case $m=2$ than for other values of $m$, uniform convergence of the sequence $\{P_n[\phi(x)]\}$ to $f(x)$ is proved. When the inverse of $\phi(x)$ is multiple-valued, there is convergence to a suitably defined mean of the values of $f(x)$ corresponding to a single value of $\phi(x)$. The method of proof is the same in principle

as one which has been applied recently in discussing the particular case $\phi(x) \equiv x$.

41. Mr. J. M. Earl: *On the convergence of polynomials of approximation on an infinite interval.*

This paper deals with the approximation to a given function $f(x)$ by means of the polynomial $P_n(x)$ of degree $n$ or less determined so as to minimize $\gamma(n)$, the integral from $-\infty$ to $+\infty$ of the $m$th power of the absolute value of the error multipled by a non-negative weight function $\rho(x)$. It represents therefore a broad generalization of the problem of Hermite series, corresponding to the special case in which $\rho(x) = e^{-x^2}$ and $m = 2$. Under suitable restrictions on $f(x)$ and $\rho(x)$, it is shown that $P_n(x)$ converges uniformly to $f(x)$ on any finite interval on which $\rho(x)$ has a positive lower bound. The proof is deduced from theorems on polynomial approximation leading to the calculation of an upper bound for the integral $\gamma(n)$. A similar discussion for the interval 0 to $\infty$ can be made to depend on the above.

42. Dr. Elizabeth Carlson: *A simplified proof for the extension of Bernstein's theorem to Sturm-Liouville sums.*

By using a method similar to one used by de la Vallée Poussin in proving the corresponding theorem for trigonometric sums, the author has proved (Transactions of this Society, vol. 26) the following theorem: The maximum of the absolute value of the derivative of a Sturm-Liouville sum of order $n (n \geqq 1)$ can not exceed $npM$, where $M$ is the maximum of the absolute value of the sum itself, and $p$ is independent of $n$ and of the coefficients in the sum. In this paper, a simpler proof is given by using a well known asymptotic expression for the characteristic solution $v_k(x)$ of a Sturm-Liouville system, in the form $v_k(x) = \cos kx + (1/k)\beta(x)\sin kx + (1/k^2) \propto (xk)$.

43. Mr. T. S. Peterson: *A class of invariant functionals of quadratic functional forms.*

In his paper *Invariant functionals of functional forms* (abstract in this Bulletin, vol. 34 (1928), pp. 8–9), A. D. Michal considered the law of transformation of a few important invariant functionals of a quadratic functional form. In particular, there were considered the Fredholm determinant and the resolvent kernel of the functional coefficients of the form. The present paper makes a detailed study of the functional invariants arising out of the Fredholm minors of the Fredholm determinant. Use is made of a theorem of Platrier to obtain the law of transformation of these invariants.

44. Professor R. L. Moore: *Concerning upper semi-continuous collections.*

A bounded continuous curve $M$ will be called a *cactoid* (*opuntioid*) if every non-degenerate maximal cyclic subset (see. G. T. Whyburn, Pro-

ceedings of the National Academy, vol. 13, pp. 31-38) of $M$ is a simple closed surface and no point of $M$ lies in a bounded complementary domain of any subset of $M$. A cactoid $M$ is said to be *aspiculate* if it does not contain any arc $t$ such that no point of $t$ except its end points is a limit point of $M-t$. An aspiculate cactoid $M$ is *simple* if no two cut points of $M$ are separated in $M$ by infinitely many different points. If $G$ is an upper semi-continuous collection of mutually exclusive continua filling up a spherical surface $S$ then the space whose elements are the continua of $G$ is, in the sense indicated in the author's paper *Concerning upper semi-continuous collections of continua* (Transactions of this Society, vol. 27, pp. 416–428), topologically equivalent to a cactoid; and, conversely, every cactoid bears this relationship to some $G$ and $S$. This theorem remains true if "cactoid" is replaced by "simple aspiculate cactoid" and the condition is imposed that the non-degenerate continua of $G$ form a contracting sequence.

45. Professor K. P. Williams: *The symbolic development of the disturbing function.*

The paper gives a modification of Newcomb's symbolic development of the disturbing function. A set of fundamental operators are isolated and lead directly to Newcomb's operators. The paper will appear in the American Journal of Mathematics.

46. Professor K. P. Williams: *A comment on certain equations in the theory of radiative equilibrium.*

In Eddington's discussion of the internal constitution of a star a set of equations occurs, which determine the coefficients in an expansion in terms of Legendre polynomials. In this paper it is shown that the equations do not warrant the conclusion that Eddington draws from a hasty examination. An integrable case is assumed, and the results confirm the conclusion of the general discussion. The paper appeared in the May number of the Astrophysical Journal.

47. Professor L. M. Graves: *The second variation for discontinuous solutions in the calculus of variations.*

In a paper presented to the Society (Sept., 1927) the author treated the extension of the Jacobi-Carathéodory conditions for discontinuous solutions to space problems, and gave a new formulation and new proof of the Carathéodory conditions which must hold at the corners. The present paper treats the same condition by means of the second variation. A new definition of conjugate point is formulated to include all cases, and the Jacobi-Carathéodory condition is that no minimizing arc can contain a pair of conjugate points. This method takes care of the exceptional cases of the previous methods, and proves also that no minimizing arc can contain a cusp. The equivalence of this formulation of the condition with the earlier one is shown.

48. Mr. E. R. C. Miles: *Fredholm solution of a generalized Neumann problem in the plane.*

Consider the class of functions expressible as potentials of a single layer, $V(M)=\int \log\,(1/MP)d\mu(s_p)$ where $\mu(s_p)$ is of limited variation on the curve $C$. The normal derivative of $u(m)$ is discontinuous across $C$, but wherever $\mu'$ exists, formulas analogous to the usual ones, but in terms of Stieltjes integrals, are obtained, if $C$ has a curvature at the point. The generalized Neumann problem states that $(1+\lambda)/(2\lambda)$ times the total flux across a curve which approaches an arc $AB$ of $C$ uniformly from inside minus $(1-\lambda)2\lambda$ times the corresponding outside limit shall be a given function $g(B)-g(A)$, of limited variation and with regular discontinuities on $C$. This problem has a unique solution provided $\lambda$ is not one of a set of characteristic values. The value $\lambda=-1$ yields a Neumann problem for the exterior region, and is not a characteristic value; $\lambda=+1$ yields a Neumann problem for the interior region, and is a characteristic value. In the latter case a necessary and sufficient condition for a solution is that $\int_C dg(s)=g(C)=0$, and the solution is determined except for an arbitrary constant.

49. Mr. E. R. C. Miles: *Fredholm solution of a generalized Dirichlet problem in the plane.*

Analogous results to those of the previous paper, with reference to the generalized Dirichlet problem, are obtained for the class of functions expressible as potentials of a double layer, $u(m)=\int[(\cos nr)/r]dv(s_p)$. Both problems are handled by solving Stieltjes integral equations, and include the usual treatments as particular cases. One interest in these problems is the fact that the methods used are extensible to the three-dimensional situation, as will be shown in a paper which the author is writing in conjunction with Professor Evans.

M. H. INGRAHAM,
*Associate Secretary.*

## THE APRIL MEETING IN NEW YORK

The two hundred sixtieth regular meeting of the Society was held at Columbia University on Friday and Saturday, April 6 and 7, 1928. The attendance included several distinguished guests (Professors Paul Alexandroff, Moscow; C. Carathéodory, Munich; and Heinz Hopf, Berlin), as well as the following one hundred twenty-four members of the Society:

C. R. Adams, R. B. Adams, Alexander, R. G. Archibald, J. W. Arnold, A. A. Bennett, Benton, William James Berry, William Johnston Berry, Blumberg, Blumenthal, Bowden, B. H. Camp, W. B. Campbell, Caris, Carrié, E. T. Carroll, Cheney, Chittenden, Alonzo Church, Coble, Abraham Cohen, L. W. Cohen, L. P. Copeland, Currier, Curry, Dantzig, Derby, Dorweiler, Dresden, Esty, Fiske, Fite, Fithian, D. A. Flanders, Frink, Gehman, Gill, Gronwall, Hall, Robert Henderson, Hickson, L. S. Hill, Hille, Himwich, Hofmann, Hollcroft, Hotelling, Huber, Jeffery, Joffe, M. I. Johnson, R. A. Johnson, Johnston, Kasner, Keller, Kline, Knebelman, Mark Kormes, Langman, Lefschetz, Lepeshkin, Littauer, Lubin, MacColl, MacInnes, H. A. Merrill, Merriman, H. H. Mitchell, T. W. Moore, Richard Morris, D. S. Morse, Marston Morse, Moyle, Murnaghan, F. H. Murray, Musselman, Neelley, C. A. Nelson, Olds, Ore, F. W. Owens, H. B. Owens, Pepper, Pfeiffer, Phalen, Pixley, Poritsky, Post, R. G. Putnam, Raudenbush, Reddick, C. N. Reynolds, R. G. D. Richardson, Ritt, Roos, Rutt, Seely, Serghiesco, Sheffer, Siceloff, Simons, Singer, Slotnick, Smail, C. E. Smith, P. A. Smith, W. M. Smith, Sosnow, Swingle, Tamarkin, J. M. Thomas, T. Y. Thomas, Veblen, Weisner, H. S. White, W. M. Whyburn, Wiener, R. L. Wilder, W. A. Wilson, Mabel M. Young, Zariski, Zippin.

On recommendation of the Council, the Society adopted amendments to the By-Laws by which the editorial committee of the Colloquium Publications become members of the Council of the Society, on the same basis as the editorial committees of the Bulletin and the Transactions, and as the Society's representatives on the editorial committee of the American Journal of Mathematics.

The afternoon sessions on Friday and Saturday were devoted to a Symposium on Analysis Situs; they were presided over by Ex-President Oswald Veblen, relieved by Professor H. H. Mitchell at the opening of the Friday session.

The following papers were presented: Friday, April 6: *Three-dimensional manifolds; generalized Riemann surfaces and knots*, by Professor J. W. Alexander; *The critical points of functions and the calculus of variations in the large*, by Professor Marston Morse; *Applications of analysis situs to algebraic surfaces, and the classification of curves on surfaces*, by Professor Solomon Lefschetz; Saturday, April 7: *Abstract sets from the point of view of analysis situs*, by Professor E. W. Chittenden; *The notion of curve and the problems that it presents*, by Professor J. R. Kline; *Application of analysis situs to the general theory of topological spaces*, by Professor P. S. Alexandroff. The discussion was opened on Friday by Professor Heinz Hopf, and on Saturday by Professor H. M. Gehman; on both days it was carried on for a considerable time and in a lively manner by several participants, among them Professor Carathéodory. The success of this symposium fully justifies the belief that there is a gain in devoting a part of each meeting to a more concentrated field of interest than can be supplied by the usual programs of short papers.

The Friday and Saturday morning sessions were sectional: Friday morning, Section A, Analysis, Professor A. A. Bennett presiding; and Section B, Geometry, Professors T. R. Hollcroft and F. W. Owens presiding; Saturday morning, Section A, Analysis Situs and Point Sets, Professor J. R. Kline presiding, and Section B, Algebra and Applied Mathematics, Professor A. B. Coble presiding. Titles and abstracts of the papers read at the sectional sessions follow below. The papers numbered 1 to 11 were read before the Section of Analysis; Nos. 12–19 before the Section of Geometry; Nos. 20–34 before the Section of Analysis Situs and Point Sets; Nos. 35–44 before the Section of Algebra and Applied Mathematics. The papers of Adams (second paper), Ayres, Blumberg (second paper), Garver, Graustein, Gronwall, Hurwitz, Miller, Moore (second paper), Stetson, Swingle, and Whyburn were read by title. Mr. Kaplan was introduced by Professor F. D. Murnaghan, and Mr. Miller by Professor R. L. Wilder.

1. Mr. A. O. Hickson: *An application of the calculus of variations to boundary value problems.*

This paper considers the boundary value problem which arises when one attempts to minimize the second variation $I_2(\eta)$ of the problem of the calculus of variations for a space of $n+1$ dimensions. A certain succession of classes of admissible arcs is chosen and the problems of minimizing $I_2(\eta)$ in these classes are considered. For each minimizing problem there is obtained the Euler-Lagrange system of ordinary linear differential equations of the second order containing a parameter $\lambda$ as a multiplier. In each class of arcs $I_2(\eta)$ has a minimum value, and these values form an increasing sequence $\lambda_0, \lambda_1, \cdots, \lambda_i, \cdots$. Also in each class there exists a set of linearly independent arcs, which give $I_2(\eta)$ its minimum value in the class considered, and which satisfy the corresponding boundary value problem with the parameter $\lambda$ equal to this minimum of $I_2(\eta)$. The results are obtained by means of well known theorems from the calculus of variations and the theory of ordinary linear differential equations.

2. Dr. G. M. Merriman (National Research Fellow): *On the expansion of functions in terms of orthogonal polynomials.* Preliminary report.

It is shown here, as an initial result in the field indicated in the title, that a function $f(x,y)$, harmonic in the interior of a closed, rectifiable Jordan curve and continuous in the closed region consisting of the curve and its interior, can be expanded in terms of a set of harmonic polynomials, normalized and orthogonalized with respect to the curve, the expansion in series being uniformly convergent interior to the curve. A set of such polynomials, which depend only on the curve, is exhibited.

3. Dr. C. F. Roos (National Research Fellow): *Solutions of limited variation of Volterra integral equations of the first kind.*

Volterra has found a continuous solution of the Volterra integral equation of the first kind, $v(x)=\int_a^x K(x,s)u(s)ds$, by reducing the equation to one of the second kind by a differentiation. W. H. Young has extended this theory to include Lebesgue integrals and has obtained a bounded solution. Both Volterra and Young have employed a differentiation to reduce the equation to a Volterra equation of the second kind. In the present paper the author shows that an integration by parts can be used to reduce the equation to the type $v(x)=u(x)\int_a^x K(x,t)dt-\int_a^x[\int_a^x K(x,t)dt]du(s)$; this integration by parts only requires that the kernel $K(x,s)$ be integrable in $s$ and of limited variation in $x$ for all $s$ and that $u(s)$ be of limited variation. Under these hypotheses and the added hypothesis that $v(x)$ be of limited variation, it is possible to obtain a solution of limited variation in the form resulting from the integration by parts.

4. Mr. H. H. Pixley: *Discontinuous solutions in the problem of depreciation and replacement.*

In a recent paper on the calculus of variations, Dr. C. F. Roos has treated a Lagrange problem with variable end points and discontinuous integrand to enable him to develop his new dynamical theory of depreciation and replacement. He assumes that the price-time and rate of production-time curves are continuous at the time of replacement. In an actual problem this would probably not be true. In the present paper a general theory corresponding to that of Roos without the assumption of continuity at the time of replacement is developed. In particular a special problem is discussed in which the second machine is started at a time and rate different from those at which the first machine stops.

5. Professor C. R. Adams: *On the linear ordinary q-difference equation.*

The equation considered is (1): $\sum_{i=0}^{n} a_i(x) f(q^n x)$, in which the known functions $a_i(x)$ are analytic or have poles at $x=0$, $q$ is a constant $\neq 0$, and $f(x)$ is to be determined. Without loss of generality we may assume $a_i(x) = a_{i0} + a_{i1}x + \cdots$ (for $|x| < R$), with at least one of the $a_{i0}$ not zero. These $a_{i0}$ are the coefficients of the characteristic equations of (1). We exhibit formal power series satisfying (1) in all possible cases. When the roots of the characteristic equation are finite and not zero, and $|q| \neq 1$, we prove directly that these solutions have positive radii of convergence; their existence and convergence can be inferred from the functional equations derived by Grévy (Paris thesis, 1894; also Annales de l'Ecole Normale, (3), vol. 11 (1894), pp. 249-323), but the present treatment possesses advantages of directness, simplicity, and brevity. When the characteristic equation has infinite or zero roots or both and $|q| \neq 1$, some (never all) of the formal series may converge only for $x=0$. At $x=\infty$ the situation is similar if the $a_i(x)$ are analytic or have poles there. The non-homogeneous equation is also treated, with similar results.

6. Professor C. R. Adams: *Note on factorial series in two variables.*

This paper appears in full in the present issue of this Bulletin.

7. Dr. Stefan Serghiesco: *On a certain double integral.*

In an earlier paper, the author has introduced a new general formula giving the number of roots of a system of equations which contains differential invariants and is represented, in the case of two equations, by a difference of two integrals, a simple and a double integral. An abstract of this paper has appeared in the Comptes Rendus for January 23, 1928. In the present paper the author makes a further study of this problem and introduces a double integral which by itself gives the number of roots of the system. This integral is then extended to $n$ equations. The treatment, through differential invariants, of this problem is thus widely generalized for the case of simple common roots of a system of equations.

8. Professors Einar Hille and J. D. Tamarkin: *Contributions to the theory of integral equations.* Preliminary communication.

Assuming the kernel to be integrable together with its square, the authors apply the theory of linear equations in infinitely many unknowns to a study of the characteristic values of a linear integral equation. These values are obtained as the zeros of an infinite determinant which is frequently easier to handle than the (modified) Fredholm determinant. Estimates are obtained for the rate of growth of the determinant and of the characteristic values in the general case as well as in various special cases, e.g., when the kernel is analytic in one or both variables or possesses a certain number of partial derivatives or satisfies a Lipschitz condition, etc. Extensive use is made of the theory of trigonometric and polynomial approximation in two dimensions.

9. Professor Harold Hotelling: *Stability in competition.*

Cournot's solution of the problem of duopoly was criticized by J. Bertrand and F. Y. Edgeworth on the ground that his "equilibrium" is unstable. It does not seen to have been recognized, however, by any writer on the subject that this instability is due to the tacit assumption that the competitors are equally accessible and their products are of equal utility, not merely to the market as a whole but to every consumer. Discarding the last condition, which is artificial, a simple example typical of actual conditions is set up and its stability is demonstrated. By means of the same example there become evident a number of qualitative properties of competitive situations generally overlooked. Among these are are some which have a bearing on the "cement trust" controversy.

10. Professor Norbert Wiener: *Coherency matrices.*

The theory of harmonic analysis already developed by the author is extended to a set of several simultaneous functions, and yields an Hermitian function of the frequency, the coherency matrix. The problem, for example, of characterizing light with respect to its state of polarization may be reduced to that of transforming a "coherency matrix" to diagonal form by a Weyl "Unitärtransformation." The theory has important applications to quantum theory and the Heisenberg matrices.

11. Dr. T. H. Gronwall: *On the convergence region of a power series in several variables.*

Fabry has shown that the region of convergence of a power series in $x$ and $y$ becomes a convex region when $\log|x|$ and $\log|x|$ are used as coordinates, and that this necessary condition on a convergence region is also sufficient. Hartogs and Faber have given simplified proofs, and the latter has extended the theorem to $n$ variables. The present note gives a proof (for $n$ variables) which is considerably simpler than the previous ones.

12. Professor J. H. Neelley: *Concerning covariants of the rational plane quartic curve with compound singularities.* First paper.

This paper discusses covariant forms of the rational plane quartic curve with ramphoid cusp. Some covariant curves degenerate, and groups have a common characteristic. The curve $R_2^4$ is taken parametrically as well as in ternary form. Some forms, which vanish identically for $R_2^4$ with this compound singularity and with other singularities, are considered. Some facts previously derived from invariants are verified by covariants.

13. Professor T. R. Hollcroft: *The postulation of multiple curves and surfaces on varieties in i dimensions.*

Numerical formulas are obtained for the postulation of a multiple line and of a multiple plane on a variety of given order in $i$ dimensions. That for a multiple line is a generalization of Noether's formula for the postulation of a multiple line on a surface. There is also found the reduction in postulation due to an intersection of two multiple lines (planes) when both lines (planes) are of the same or different multiplicities on a variety of given order in $i$ dimensions. Numerical formulas for the postulation of any given curve or surface of given multiplicity on a variety of given order in $i$ dimensions are derived from the above. The postulation of any given one- or two-dimensional manifold of given multiplicity on a variety of given order is similarly obtained.

14. Professor H. R. Phalen: *Metric properties of the quadric of Moutard.*

Under the assumption that a certain surface $S$ is non-developable. this paper defines a quadric of Moutard, derives its equation, and obtains the discriminant and the coefficients of the discriminating cubic, the coordinates of the center, and the mean and total curvature of $S$ and of the Moutard quadric at the origin, and enunciates the following theorems: (a) At a given point $P$ upon $S$ the Moutard quadric and the surface $S$ have the same total curvature, the same mean curvature, and the same surface normal. (b) The center of every Moutard quadric at a point $P$ upon $S$ lies outside the tangent plane to $S$ at $P$. (c) At a point $P$ upon $S$, twelve tangents may be drawn such that the corresponding Moutard quadric is non-central. (d) If the surface $S$ be defined by an equation of the form $z=f(x,y)$ in which no term of the right member is of degree less than the third, the Moutard quadric degenerates into the tangent plane counted twice. (e) If in the equation of the Moutard quadric the term in $z^2$ is absent, the quadric is central. In conclusion, criteria are deduced for classifying the various quadrics.

15. Professor Edward Kasner: *Characterization of geodesic families.*

It is shown that just as natural families may be characterized by integrability with reference to orthogonality (the author's converse of

the Thomson-Tait theorem, 1909), so geodesic families of curves may be characterized geometrically by integrability with reference to any polarity (the base cone of the polarity may vary arbitrarily from point to point). Polar transversalities are the only projective correspondences which can serve as transversality relations (see these Transactions, July, 1928). The results are valid in space of three or more dimensions.

16. Professor Edward Kasner: *The second derivative of a polygenic function.*

The second derivative $\sigma=d^2w/dz^2$ of an arbitrary polygenic function $w=\phi(x,y)+i\psi(x,y)$ depends not only on the point $z=x+iy$ but also on the slope and curvature of the path of approach, that is, $\sigma=\Omega(x,y,y',y'')$. For a given $y'$ the locus of $\sigma$ in its complex plane is a straight line. As $y'$ varies this line envelopes a cardioid. If $y''=0$ (rectilinear second derivative) the locus of $\sigma$ is a limaçon. For a complete representation it is necessary to parametrize these loci (clocked cardioid, analogous to clocked circle, see Science, December 16, 1927). Higher derivatives are also discussed. This paper has been offered to the Transactions.

17. Professor W. C. Graustein: *On the average number of sides of polygons of a net.*

The leading theorem in this paper, stated in its simplest form, reads as follows: If in a net of regular hexagons covering the plane a finite simply connected set of hexagons is removed and the region thus vacated is covered by polygons in such a way that the number of sides issuing from each vertex in the net remains three, then the average number of sides, per polygon, of the polygons in the net remains six. Similar theorems hold for nets of squares and equilateral triangles.

18. Professor W. C. Graustein: *Note on a certain type of parabola.*

The author discusses certain striking properties of a parabola which is tangent to the line at infinity in a circular point.

19. Dr. J. M. Stetson: *Theorems of permutability.*

It seems to have escaped notice that a part of the classic theory of Levy transformations may be restated thus: the transformations of Laplace and Levy and the inverse of the transformations of Levy are permutable. From this follows the fundamental theorem: Any type of transformation made by combining these three is permutable, and is permutable with any other such type. This general theorem appears to include as special cases all known theorems regarding permutability of transformations of conjugate nets in which no restrictions are made as to the character of the nets, if we note that either radial transformations, or parallel transformations, but not both, may be added to the list of transformations given in the theorem. We obtain a simple proof of the permutability of transformations $F$, and of the transformations giving derived nets (and their inverses), a fact which seems to be new except for the special case of reciprocal derived nets.

20. Professor R. L. Wilder: *On connected and regular point sets.*

This paper will appear in full in an early issue of this Bulletin.

21. Professor R. L. Wilder: *Concerning irreducibly connected sets and irreducible regular connexes.*

If the point set $M$ be irreducibly connected about a set $K$ but not about any proper subset of $K$, $K$ is called a basic set about which $M$ is irreducibly connected. Among the results obtained are the following: (1) If $K$ is a basic set about which $M$ is irreducibly connected, then (a) $K$ is identical with the set of non-cut points of $M$ and (b) if $K$ consists of $n$ points, the number of cut points of $M$ of order greater than one is at most $n-2$. (A point $P$ is called a cut point of order $n$ if $M-P$ is the sum of $n$ mutually separated sets.) Furthermore, if $P$ is a point of $M$ such that $M-P$ is the sum of $n$ components $M(i=1, 2, \cdots, n)$, and if $M_i \times K = K_i$, then the set $K_i+P$ is a basic set about which $M_i+P$ is irreducibly connected. (2) If $M$ is closed and bounded, then (a) $M$ contains a basic set about which it is irreducibly connected; (b) $M$ contains at least two non-cut points. Special attention is paid to the case where $K$ consists of a finite number of points as well as to the case where $M$ is regular.

22. Professor R. L. Wilder: *Concerning the relation of sets $F_{\sigma\epsilon}$ to their complements in euclidean space of n dimensions.*

We denote by $F_{\sigma\epsilon}$ a set $M$ which is the sum of a denumerable collection of mutually exclusive, closed and bounded sets $M_1, M_2, \cdots$, whose diameters converge to zero as $n$ tends to infinity. Among the results obtained are the following: (1) If $D$ is a connected domain in euclidean space $E_n$ such that no set $M_n$ contains points of both $D$ and $E_n-D$, and if no set $M_n$ separates $E_n$, then $D-M\times D$ is a properly non-vacuous, arcwise connected point set. (2) Let $A$ and $B$ be points of a bounded connected domain $D$ in $E_n$ such that $M$ is a subset of $D$ and no set $M_n$ separates $A$ from $B$ in $D$. Then $D-M$ contains an arc from $A$ to $B$. (3) If $M$ lies in $E_2$, no $M_n$ separates $E_2$, $K$ is a component of a set $M_n$, and $e$ is any positive number, there exists a simple closed curve $J$ such that every point of $J$ is at a distance from $K$ less then $e$ and such that $J$ and $M$ have no point in common.

23. Mr. L. W. Cohen: *On sets homeomorphic with linear sets.*

Definitions of closed arcs, half open arcs, and open arcs are stated for a separable metric space, and such sets are shown to be homeomorphic respectively with closed, half open, and open intervals on the linear continuum. The following theorem is proved: Necessary and sufficient conditions that a set $M$ in a separable metric space be homeomorphic with a linear set are that (1) the components of $M$ are points of arcs; (2) if $p$ is a point component of $M$, then $\dim_p M=0$; (3) if $a$ is an end point of an arc component $T$ in $M$, then $\dim_a(M-T+a)=0$; (4) if $p$ is a point of an arc component $T$ distinct from its end points, then $p$ is not a limit point of $M-T$. In this theorem an arc is either closed, half open, or open.

24. Dr. P. A. Smith: *The regular components of a surface transformation.*

Regular components are maximal connected regions of "uniform approach" toward certain invariant limit sets on the surface. These components have been studied by Birkhoff and Smith in a paper to appear in the Journal de Mathématiques, but only on surfaces of genus zero. The results are here extended to the general case.

25. Professor E. W. Chittenden: *On covering theorems in general topological space.*

A family $F$ of sets $G$ of elements of a class $P$ is said to be a *covering* of a subset $E$ of $P$ and to be of *type* $T$ in case each element $p$ of the set $E$ belongs to a set $G$ of $F$ and is also in the relation $T$ to some set $G$ of $F$. The undefined relation $T$ is assumed to be definite between elements and subsets of $P$. For example, in a topological space the relation $T$ may be defined to mean that the element $p$ is interior to $G$. A covering $F$ is called reducible provided it contains a covering of the same type and of lower cardinal number. The covering theorems of Borel, Lebesgue, and Lindelöf, together with certain extensions suggested by Alexandroff, Urysohn, and Hildebrandt, are considered in relation to this concept of reducibility. In particular, the paper contains sets of necessary and sufficient conditions for the reducibility of every covering $F$(of a set $E$) which is of type $T$. The results obtained are applied to the theory of topological spaces in general.

26. Dr. P. A. Smith: *Transformations of the boundary of a simply connected invariant region.*

This paper is a report on certain questions which arise in a topological study of the structure of surface transformations.

27. Professor Henry Blumberg: *On various metric properties of general sets and functions.*

In this paper the following theorems are proved: If $S$ is a point set in euclidean $n$-space, it may be uniquely represented (sets of measure zero regarded as negligible) in the form $S=M+N$, where $M$ is measurable and $N$ is such that at every point of $N$ the exterior metric density of $N$ and also that of the complement of $N$ is equal to 1. If $A$ and $B$ are any two sets whatever, overlapping or not, the exterior metric density of $A$ is 0 or 1 at every point of $B$, with the possible exception of a set of measure zero. If $f(x)=f(x_1, \cdots, x_n)$ is an arbitrary real function and $u_m(x)$, $l_m(x)$ represent the upper and lower metric limits of $f$ at $x$, that is, the largest and smallest values of $\eta$ for which $(x, \eta)=(x_1, x_2, \cdots, x_n, \eta)$ is "approached" but not "infinitesimally approached," in the metrical sense, by the surface $y=f(x)$, then $u_m(x)$, $l_m(x)$ are "metrically upper-semi-continuous," and, moreover, approximately continuous, in the sense of Denjoy, almost everywhere; hence $u_m(x)$ and $l_m(x)$ are measurable. Therefore, given any function $f(x)$, there exist two measurable functions $m_1(x)$ and $m_2(x)$ such that

$m_1(x) \geqq f(x) \geqq m_2(x)$ almost everywhere, and the metrical approach of $y=f(x)$ at $(x, m_1(x))$ as well as at $(x, m_2(x))$ is "full," in the sense of metric density.

28. Professor Henry Blumberg: *Note concerning the inductive principle.*

It seems not to have been remarked that we may formulate the inductive principle for all linearly ordered sets as follows, and thereby comprehend in our formulation the particular applications to the normally ordered sets and to the linear continuum: If the property $P$ holds for all the elements of some segment $S$ of a linearly ordered set $A$, and if the validity of $P$ for all the elements of a segment $S$ of $A$ implies its validity for all the elements of some proper supersegment $S'$ of $A$, then $P$ holds for all the elements of $A$. The principle is extensible to multiply ordered sets.

29. Professor H. M. Gehman: *Concerning certain types of non-cut points, with an application to continuous curves.*

A point $P$ of a continuum $M$ is called a *point of type* 1, if given any positive number $\epsilon$, there exists an $M$-domain containing $P$ of diameter less than $\epsilon$ whose complement is connected. Similarly, $P$ is a *point of type* 2, if there exists an open set in $M$ containing $P$ of diameter less than $\epsilon$ whose complement is connected. A point of type 1 is a point of type 2, and a point of type 2 is a cut point, but not conversely, in either case. Certain properties of points of types 1 and 2 are proved in this paper, and an application is found in the proof of the following theorem: If $M$ is a continuous curve (bounded or unbounded) in space of any finite number of dimensions, and if $P$ is a non-cut point of $M$, then given any positive number $\epsilon$, there exists an $M$-domain containing $P$ of diameter less than $\epsilon$ whose complement is connected; if $P$ is a cut point of $M$, then given any positive number $\epsilon$, there exists an $M$-domain containing $P$ of diameter less than $\epsilon$ whose complement consists of a *finite* number of maximal connected sets.

30. Professor G. T. Whyburn: *Concerning certain accessible points of plane continua.*

Let $K$ denote the set of all those points of a plane continuum $M$ each of which is accessible from at least two complementary domains of $M$. Then (1) if $H$ is any uncountable subset of $K$, there exist points $X$ and $Y$ of $H$, an uncountable subset $E$ of $H$, complementary domains $R_1$ and $R_2$ of $M$, and two continua $L$ and $N$ such that (a) $L \cdot N = X + Y$, (b) $L + N = M$, (c) every point of $E$ is a cut point of $N$ and is accessible from both $R_1$ and $R_2$; (2) all, save possibly a countable number, of the points of $K$ are points of Menger order two of $M$; (3) if $G$ is the collection of all the finite subsets of $M$ each of which is an irreducible cutting of $M$, and $T$ is the sum of all the point sets of the collection $G$, then all, save possibly a countable number, of the points of $T$ are points of Menger order two of $M$. An interesting generalization of (1) is given which can be applied in the study of the relation between a subcontinuum of a plane continuous curve and its complement in that curve.

31. Professor G. T. Whyburn: *A generalized notion of accessibility.*

A continuum $K$ will be said to be accessible from a point set $R$ provided that if $A$ is any point of $R$, a bounded continuum exists which contains both $A$ and $K$ and is a subset of $R+K$. The following results are established. (1) If $R_1$, $R_2$, and $R_3$ are mutually exclusive connected point sets in a plane $S$, there do not exist, in $S-(R_1+R_2+R_3)$, three mutually exclusive continua each of whch is accessible from both $R_1$ and $R_2$ and contains a limit point of $R_3$. (2) Every component of the $M$-boundary of every connected open subset $R$ of a continuous curve $M$ (in $n$ dimensions) is *regularly accessible* from $R$. (3) For open subsets of a continuous curve, accessibility by continua as above defined is equivalent to accessibility by a special kind of a continuum, which is an "ideal arc" all of whose elements are points except possibly one. (4) The following properties are equivalent for continua $M$ (in $n$ dimensions): ($\alpha$) every subcontinuum of $M$ is a continuous curve, ($\beta$) every connected open subset of $M$ has property $S$, and ($\gamma$) every point of the $M$-boundary of every connected open subset $R$ of $M$ is regularly accessible from $R$.

32. Mr. P. M. Swingle: *A certain type of continuous curve and related point sets.*

This paper is a study of continuous curves every subcontinuum of which is itself a continuous curve; special sections are devoted to (1) continua irreducible between two points, and (2) sets which are composed of the points in sets of arcs joining two distinct closed sets, with a view to determining the relation between the number of possible distinct arcs and the number of common points. The following conditions are each shown to be necessary and sufficient that every subcontinuum of a continuous curve be itself a continuous curve: (1) If $P$, $A$ and $B$ are any three points of a subcontinuum $N$ of $M$ such that there exists no point, distinct from $P$, contained in every arc of $N$ from $P$ to $A+B$, then there is an arc $APB$ in $N$. (2) If $N$ is a connected subset of $M$, $A$ and $B$ are distinct points of $N$, $L$ is the set consisting of the points of arcs $AB$ of $N'$, and $Q=L\times N$, then $Q$ is connected. (3) If $L$ is the set consisting of points of a set of arcs of $M$ joining two distinct closed sets $E$ and $F$, then every point of $L'$ is contained in an arc of $L'$ joining $E$ and $F$. It is also shown that in order that every subcontinuum of a continuum $M$ be a continuous curve it is sufficient that if $A$ and $B$ are any two distinct points of $M$ there exist a finite number of points which separate $A$ and $B$ in $M$.

33. Mr. E. W. Miller: *Note on the relation between component and quasi-component in a closed point set.*

In a metric space, if $M$ is a compact and closed set and $P$ is a point of $M$, then the quasi-component of $M$ determined by $P$ and the component of $M$ determined by $P$ are identical. If the condition of compactness is removed this is no longer true. However, in a metric space every bounded subset of which is compact, it is shown that if $M$ is a closed set then in

order that the quasi-component of $M$ determined by $P$ should be identical with the component of $M$ determined by $P$ it is sufficient that the component be compact.

34. Dr. W. L. Ayres (National Research Fellow): *On continua which are disconnected by the omission of any point.*

There exists a plane continuum $M$ such that (1) every point of $M$ is a cut point of $M$, (2) every continuum $K$ such that every point of $K$ is a cut point of $K$ is homeomorphic with some subset of $M$. In order that every point of a plane continuum $M$ be a cut point of $M$ either of the following conditions is necessary and sufficient: (1) every point of $M$ be accessible from two unbounded complementary domains, (2) $M$ be non-dense, the boundary of every complementary domain be an open curve and only a finite number of complementary domains have points within any circle. If $P$ is a non-cut point of a continuous curve $M$ in $n$ dimensions and $\epsilon$ is any positive number, there exists a continuous curve $N$ and a positive number $\delta$ such that (1) $M$ contains $N$, (2) the diameter of $N$ is less than $\epsilon$, (3) every point of $M$ whose distance from $P$ is less than $\delta$ belongs to $N$, (4) $M-N$ is connected.

35. Dr. T. W. Moore (National Research Fellow): *Extended results in elimination.*

This paper presents a means of elimination for double binary forms in sets of digredient variables. The fundamental problem is to get the condition that three such forms have a common solution expressed as a determinant involving no extraneous factors and not vanishing identically. The method employed is an extension of that used by Morley and Coble for forms in one set of variables published in their article *New results in elimination*, in the American Journal of Mathematics, vol. 49 (1927), No. 4. A covariant containing the coefficients of all three forms and vanishing when they have a common solution is used to obtain this determinant representation. The method has already been applied to a limited range of cases and appears capable of extension to forms containing other than binary variables.

36. Dr. T. W. Moore: *On the complete system of combinants of two binary quintics.*

This paper contains a system of combinants of two binary quintic forms, derived as the simultaneous system of covariants of their first and third transvectants, an octavic and a quartic respectively. It is a continuation of a paper presented to this Society in February, 1927 (see this Bulletin, vol. 33, p. 260) in which the case of the invariant combinants only was considered. The list of forms found in the system contains 113 members. All these are equivalent to the 69 independent forms in the complete system of the binary octavic apolar to the two given quintics, but the exact connection has not yet been established.

37. Professor J. R. Musselman: *A planar imprimitive group of order* 216.

In a recent paper in the American Journal, the existence of a planar imprimitive group of order 216 connected with four Hesse configurations was shown by the writer. The purpose of the present paper is to discuss the group and its associated geometry in detail. Of special interest is a configuration of twelve triangles. A new canonical form used in discussing four-fold projective triangles leads to a ruler construction for them and a compass construction for a Clebsch six-point. The group in $S_n$ is of order $6^n(n+1)!$.

38. Professor Oystein Ore: *Arithmetical theory of Galois fields.* Second communication.

In the author's first communication on this subject it was shown that the complete system of residues for prime-ideal powers can be derived by successive adjunctions of roots of binomial and trinomial congruences, corresponding to the different subgroups of the "Zerlegungs"-group of Hilbert. In the present paper various consequences of these results are derived, and it is proved that the normal congruences are either all binomial or all trinomial. It then follows that in the binomial cases both the "Trägheits"-group and the "Verzweigungs"-group of Hilbert must be cyclic groups.

39. Dr. Raymond Garver: *A new normal form for quartic equations.*

This paper has appeared in the May-June issue of this Bulletin.

40. Dr. Raymond Garver: *The Tschirnhaus transformation.*

In this paper, which is to appear in the Annals of Mathematics, the author makes what he thinks is the first attempt to collect and consider the various results that have been obtained by the use of the Tschirnhaus transformation. A bibliography of 100 titles accompanies the paper.

41. Professor F. D. Murnaghan: *The stress components in an elastic solid subjected to a finite deformation.*

When an elastic solid is subjected to a finite strain it is shown that the stress components are derivable from the elastic energy $\phi$, which is assumed to be a function of the six strain components $(e_1, \cdots, e_6)$, by means of the equations $X_x=\partial\phi/\partial e_1-(2e_1\partial\phi/\partial e_1+e_5\partial\phi/\partial e_5+e_6\partial\phi/\partial e_6)$; $Y_z=\frac{1}{2}\partial\phi/\partial e_4-(2e_4\partial\phi/\partial e_2+e_3\partial\phi/\partial e_4+e_5\partial\phi/\partial e_6)$; $Z_y=\frac{1}{2}\partial\phi/\partial e_4-(2e_4\partial\phi/\partial e_3+e_2\partial\phi/\partial e_4+e_6\partial\phi/\partial e_5)$. In these expressions $e_1=\partial U/\partial x-\frac{1}{2}[(\partial U/\partial x)^2+(\partial V/\partial x)^2+(\partial W/\partial x)^2]$; $2e_4=\partial W/\partial y+\partial V/\partial z-[(\partial U/\partial y)(\partial U/\partial z)+(\partial V/\partial y)(\partial V/\partial z)+(\partial W/\partial y)(\partial W/\partial z)]$, etc., where $(U, V, W)$ is the displacement vector of any particle of the medium. If we assume that the stress tensor is symmetric, we have three equations of the type $Y_z=Z_y$, which form a complete

system, and $\phi$ must be a function of the three strain invariants. Hence for a non-isotropic, that is, crystalline medium, we must give up the assumption that the energy of deformation is a function of the strain components.

42. Professor F. D. Murnaghan and Mr. Carl Kaplan: *On the foundations of electromagnetic theory.* Preliminary communication.

An examination is made of the connection between the two fundamental covariant alternating tensors of rank two, namely the $(B, cE)$ and the $(D, -cH)$ tensors, which appear in Maxwell's equations. It appears that the usual equations $B=\mu H$ and $D=\epsilon E$ are but special cases of a more general theory which would put $B=\alpha D+\beta H$, $E=\gamma D+\delta H$, where $\alpha$, $\beta$, $\gamma$, $\delta$ are constants.

43. Dr. Hillel Poritsky (National Research Fellow): *n-dimensional potential theory.*

In this paper we indicate the elements of the theory of $p$-vector potentials in $n$-dimensional euclidean space, and generalize to $n$ dimensions some of well known propositions of three-dimensional potential theory. As an example, we obtain one chain of theorems that reduces for $n=3$ to three links of which the outer ones are equivalent to Gauss's flux theorem while the middle one is equivalent to the linkage theorem of electromagnetism: the work done in carrying a unit magnetic pole around a closed path $C$ in the field of unit electric current flowing in a circuit $\gamma$ is equal to $4\pi n$, where $n$ is the number of times $C$ links with $\gamma$.

44. Professor W. A. Hurwitz: *On Bell's arithmetic of Boolean algebra.*

This paper suggests a modification of the arithmetic of Boolean algebra given by Bell (Transactions of this Society, vol. 29, pp. 597–611) by proposing a definition of congruence which retains all the properties postulated by Bell and also restores several analogies to rational arithmetic.

ARNOLD DRESDEN,
*Associate Secretary.*

## THE APRIL MEETING OF THE SAN FRANCISCO SECTION

The fifty-fourth regular meeting of the San Francisco Section was held at Stanford University on Saturday, April 7, 1928. In the absence of the regular Chairman, Professor E. R. Hedrick presided during the early part of the meeting and Professor E. W. Brown during the latter part. The total attendance was twenty-seven, including the following twenty members of the Society:

Bernstein, Biggerstaff, Blichfeldt, Bright, E. W. Brown, Buck, Cajori, Crum, E. R. Hedrick, Hoskins, R. L. Jackson, Vern James, D. N. Lehmer, F. R. Morris, Noble, Pehrson, T. M. Putnam, Stager, A. Pell Wheeler, A. R. Williams.

The Secretary announced that the Summer meeting of the Section will be held at Reed College on June 2. It was decided to hold the next Spring meeting at Stanford University on April 6, 1929.

Titles and abstracts of papers read at the meeting follow.

1. Professor E. T. Bell: *Certain class—number relations implicit in the Nachlass of Gauss.*

This paper appears in full in the present issue of this Bulletin.

2. Professor E. T. Bell: *Remark on the number of classes of binary quadratic forms of a given negative determinant.*

The long sought means are indicated for an independent (strictly elementary) proof of the fact that, if $p$ is an odd prime, the number of quadratic residues in the set $1, 2, 3, \cdots, (p-1)/2$ exceeds the number of quadratic non-residues. This paper will be published in the Proceedings of the National Academy.

3. Professor E. T. Bell: *Remarks on the quaternary quadratic identity of Hermite.*

In a recent note in the American Mathematical Monthly, it was observed that Hermite's identity does not admit of generalization by algebraic transformations of the coefficients. The present remarks refer to unpublished work concerning the composition of Hermitian forms in 4 or 8 indeterminates.

4. Professor E. T. Bell: *Note on difference equations defining enumerative arithmetical functions.*

The trivial algebraic origin of an extensive class of definitive difference equations for enumerative numerical functions is indicated. In particular, the important identity of J. V. Ouspensky (Bulletin de l'Académie des Sciences de l'URSS, 1925, p. 647), concerning representations as sums of squares, is traced to its obvious origin in the formula for the derivative of a product. The note will be published in the Bulletin of the Calcutta Mathematical Society.

5. Professor E. T. Bell: *A generalization of circulants.*

This is a self-contained, independent detail in a recasting of the theory of algebraic numbers in terms of finite processes only. It will appear in the Proceedings of the Edinburgh Mathematical Society.

6. Professor E. T. Bell: *Ternary characteristics of primes.*

This paper has appeared in full in the May-June number of this Bulletin.

7. Professor E. T. Bell: *Certain completely solvable systems of simultaneous diophantine equations.*

This paper will appear shortly in the American Mathematical Monthly. The equations considered are of any degree in any number of indeterminates.

8. Professor E. T. Bell: *A property of resultants.*

The factorization property of the resultant of two algebraic equations, one of which is binomial, is extended to any pair of algebraic equations. The paper will appear in the Messenger of Mathematics.

9. Professor Florian Cajori: *Early history of partial differential equations and of partial differentiation and partial integration.*

This paper is a criticism of recent European expositions of the theory of fluxions, according to which partial fluxional equations are impossible. The history of partial processes and of partial differential equations is traced on the Continent to the time of Euler, and in Great Britain to the opening of the nineteenth century.

10. Professor Florian Cajori: *A revaluation of Harriot's "Artis analyticae praxis."*

John Wallis's appraisement of Harriot's algebra was strained by national bias. Conflicting estimates of later date, and unwarranted statements found in the most recent histories, induced the present author to make a re-examination of Harriot's book.

11. Professor E. R. Hedrick: *On derivatives of non-analytic functions.*

Kasner has recently shown (Proceedings of the National Academy, vol. 13 (1928), pp. 75–82) that the increment ratio $\Delta w/\Delta z$ for a non-analytic function $w=f(z)$ approaches points on the circumference of a circle as the increments approach zero. The present paper emphasizes the fact that one diameter of the Kasner circle is determined by $\partial w/\partial x$ and $\partial w/\partial(yi)$, where $z=x+yi$, and that another diameter is determined by the maximum and minimum values of $r=|\partial w/\partial z|^2$, which was discussed by Hedrick, Ingold, and Westfall (Journal de Mathematiques, vol. 2 (1923), pp. 327–342). The Tissot indicatrix modified in a manner that is not essential, discussed in that paper is in fact the polar diagram of the distances from the origin to the points of the Kasner circle. The diameter of the Kasner circle is the *ellipticity* discussed in that paper, when a nonessential modification is made. The fact that the Tissot characteristic lines are orthogonal corresponds to Kasner's theorem that orthogonal directions give points at the ends of a diameter of the Kasner circle. This paper will appear in the Proceedings of the National Academy.

12. Professor E. R. Hedrick: *On the increment-ratio for non-analytic functions.*

The increment-ratio $q=\Delta w/\Delta z=[f(z)-f(a)]/(z-a)$ for a function $w=f(z)$ may be considered as a new function of $z$. Its properties are partly known through those of its limit, $dw/dz$, which ordinarily depends upon the slope of the curve on which $\Delta z$ approaches zero. But $q=r+it$ is defined for all values of $z$ for which $f(z)$ is defined, except perhaps $z=a$. In this paper it is shown by direct calculation that the Jacobian of the transformation from $(x, y)$ to $(r, t)$ vanishes precisely on the Kasner circle. In general, the vanishing of the Jacobian of any non-analytic function defines a curve of which a special case is the branch point of an analytic function; this curve may be called the *edge of regression* on account of its similarity to the edge of regression of a developable surface. The edge of regression of the increment-ratio $q$ is then the Kasner circle, which is the envelope of curves corresponding to any pencil of curves through the point $z=a$ in the $z$ plane. The Riemann surface over the $q$ plane is, in general, two-leaved, joined along the Kasner circle. However, not every function whose edge of regression is a circle can be set up as an increment-ratio. The result of this paper will appear in the Proceedings of the National Academy; a detailed proof will appear in the Commemoration Volume (1928) of the Calcutta Mathematical Society.

13. Professor E. R. Hedrick: *Analytic points of non-analytic functions.*

The derivative of a non-analytic function $w=f(z)$, $w=u+vi$, $z=x+yi$ may exist at a point $z=a$, and it does if $u_x=v_y$, $u_y=-v_x$. Such a point will be called an *analytic point.* If the Jacobian, $J$, of $u$ and $v$ does not

vanish, $f(z)$ near $a$ is given by a finite Taylor expansion in the powers of $(z-a)$, plus a remainder, $(z-a)^n Q_n(z)$, where $Q_n(z)$ is not analytic at $z=a$. The analytic point can be "*removed*" by subtracting from $f(z)$ a polynomial and dividing by $(z-a)^n$. Any finite number of analytic points can be removed. Under obvious conditions, the expansion is infinite and $f(z)$ is analytic in a region. A limit point $l$ of such analytic points is itself an analytic point. The functions $Q_n(z)$ for $z=l$ is non-analytic at $z=l$; hence it is non-analytic except at a finite number of points near $l$. Then $f(z)$ must have the same property. It follows, under suitable hypotheses, that if $f(z)$ has more than a finite number of analytic points in a closed region, it is analytic in that region. This result generalizes considerably the usual theorems on the independence of the equations $u_x=v_y$, $u_y=-v_x$ when $J\neq 0$.

14. Professor E. R. Hedrick: *On the law of the excluded middle from the axiomatic standpoint.*

It has been assumed by recent writers that the law of the excluded middle would be sustained by those who treat mathematics axiomatically, as well as by believers in a priori validity of the law. In the present paper, it is pointed out that the words "true" and "false" should be defined in an axiomatic discussion, and that the statements "$p$ is false," "$p$ is not true (or untrue)," and "not-$p$ is true" may have different meanings. The following definitions are suggested. Given a set of axioms, a proposition $p$ is called *true* if it can be reached by a finite number of syllogisms. A *contradiction* arises if $q$ is *true* and not-$q$ is *true*. Finally, $p$ is called *false* if the axioms, together with $p$, lead to a contradiction after a finite number of syllogisms. A set of axioms that lead to a contradiction is called *inconsistent*. It follows as a theorem that if $p$ is *false* and $p$ is *true*, then the axioms are inconsistent: such sets of axioms are possible, though not desirable; to exclude them is to assert the law of contradiction. With such definitions, even if the axioms are consistent, the law of the excluded middle would be open to question, and could not be said to have a priori validity.

15. Dr. A. R. Williams: *Quintic surfaces with two coincident double lines.*

The purpose of this paper is to contrast two kinds of quintic surfaces. Both have a double curve consisting of two coincident lines. But in the first case the lines have approached coincidence remaining in the same plane, and the resulting surface is not, in general, rational. In the second case the lines, in approaching coincidence, have remained skew, and the surface is rational. Five conditions must be added to the general surface of the first type to make it of the second type.

16. Professor E. W. Brown: *Harmonic analysis of the disturbing function, with a remainder formula.*

The author shows how an approximate value for the remainder after some definite term in the expansion of the disturbing function could be

obtained, and he gives the general series from which this value can be deduced. With its use, harmonic analysis with two arguments would give the coefficients of the needed periodic terms without the extensive numerical calculations which are usually necessary.

B. A. BERNSTEIN,
*Secretary of the Section.*

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## A CORRECTION

BY W. J. TRJITZINSKY

My attention has been drawn to the fact that Theorem I of my paper, *Expansion in series of non-inverted factorials*, (this Bulletin, vol. 34 (1928), pp. 193-196), is not new. This theorem is a special case of a theorem found on page 229 of N. E. Nörlund's *Differenzenrechnung*, and is due to Nörlund; it appeared first in Annales de l'Ecole Normale Supérieure ((3), vol. 39 (1922)). The theorem is also a special case of a theorem less general than that of Nörlund, which was proved by Carlson in Nova Acta Soc. Scient. Upsaliensis ((4), vol. 4 (1915)). Neither of these was known to me at the time of the publication of my paper.

# SOME PHILOSOPHICAL ASPECTS OF MATHEMATICS*

BY ARNOLD DRESDEN

"The older I get," observed Mr. van Koppen, "the more I realize that everything depends upon what a man postulates. The rest is plain sailing." (Norman Douglas, South Wind, p. 441.)

It will be well for me to begin this paper with the remark which would be superfluous later on, that it does not presume to make a contribution to mathematics. It is about mathematics, not of them. That I have nevertheless chosen it for presentation in response to the invitation of the program committee, is a result of my conviction that it is useful to reflect from time to time on the character of the structure which is being developed by an ever increasing group of workers. For, while many parts of this structure have become familiar through long acquaintance, it is steadily reaching out in new directions, opening up vistas which stimulate the imagination to envisaging still further extensions and also such as give hitherto unsuspected views of the familiar parts. Ever greater become the distances which separate those who work at different sections of this structure and it is becoming increasingly difficult for them to keep feeling with the fundamental plan which determines its development. The day is long past when unity could be secured through the coordinating agency of a single mind, or even of a group of closely associated minds; and no one can dream of holding together the lines of communication which connect the different parts of the magnificent structure. Perhaps this state of affairs, well recognized by every one, justifies an occasional speculation on the intrinsic character

* Address presented at the request of the American Mathematical Society at the joint meeting with the Mathematical Association of America and Section A of the A. A. A. S. at Nashville on December 29, 1927.

of our science. It is to indicate their speculative character that the following remarks are collected under the present title.

Two contrasting circumstances, the one old, the other of recent origin, have been the starting point for these considerations. On the one hand, we find a persistent faith, particularly on the part of those who are not very intimately familiar with the subject, in the incontestable validity of the conclusions of mathematics. They find in it an escape from the uncertainties of a doubting world, a firm rock to which to hold on in the midst of a fluid universe. To find something which is "mathematically certain" is still, as it ever was, the desire of every seeker after "truth." It is this persisting faith which gives to the mathematician a rather unique place, which provides his subject with a characterization, apparently sufficient for the uninitiated. At the same time it should lead him to a searching through of his science, to inquire whence this faith comes, whether and why it can be accepted also by those who are initiated and left undisturbed in the minds of others.*

For, on the other hand, there has grown up among mathematicians during the last decade or two, a tendency which seems to cast doubt on a large and important part of mathematics. By some this tendency has been characterized as revolutionary, as "bolshevistic" (oh, misery of words!), as subversive of the wholesomeness which had always characterized mathematics. This tendency, inaugurated by the Dutch mathematician, L. E. J. Brouwer, has several distinct aspects. It has been discussed elsewhere† and has been referred to by Professor Pierpont in the address made at yesterday's meeting. I have but to recall to you Hilbert's animadversions upon this work,‡ to have you

* A more extended discussion of "mathematical certainty" forms the subject of a paper to be published in an early number of Scientia.

† See the author's article, *Brouwer's contributions to the foundations of mathematics*, this Bulletin, vol. 30 (1924), p. 31.

‡ See Pierpont, *Mathematical rigor, past and present*, this Bulletin, vol. 34 (1928), p. 52.

realize how serious an attack upon the citadel of our science it is held to constitute. If this is indeed the character of the position of Brouwer, it is necessary for mathematicians either to refute that position decisively, or else to abandon the part of the field that is under attack and to retire to "previously prepared positions." It is the former of these alternatives which was undertaken in a paper by Barzin and Errera, at least with reference to one important aspect of Brouwer's position,* namely, his denial of the unlimited validity for mathematical logic of the Law of the Excluded Middle. And it will also be with this aspect of Brouwer's position that we shall be concerned in this paper.†

It is well known that the Aristotelian logic which ordinarily is tacitly assumed as the basis for logical reasoning proceeds from three fundamental canons, namely, the Law of Identity, the Law of Contradiction, and the Law of the Excluded

* See M. Barzin et A. Errera, *Sur la logique de M. Brouwer*, Bulletins de la Classe des Sciences, Académie Royale de Belgique, January, 1927, p. 56.

† Although as a matter of convenience, this point of view is here ascribed to Brouwer, who is indeed its first contemporary representative and, with some of his pupils, its principal exponent, it is not his alone. For arguments on his side of the question, see H. Weyl, *Über die neue Grundlagenkrise der Mathematik*, Mathematische Zeitschrift, vol. 10 (1921), p. 39; Weyl, *Der circulus vitiosus in der heutigen Begründung der Analysis*, Jahresbericht der Vereinigung, vol. 28 (1919), p. 85; Weyl, *Randbemerkungen zu Hauptproblemen der Mathematik*, Mathematische Zeitschrift, vol. 20 (1924), (particularly p. 146–150); R. Wavre, *Y a-t-il une crise des mathématiques*, Revue de Métaphysique et de Morale, vol. 31 (1924), p. 435; also the list of references given on the last page of the article by Barzin and Errera quoted above. There is also a considerable literature on the opposing view; see J. v. Neumann, *Zur Hilbert'schen Beweistheorie*, Mathematische Zeitschrift, vol. 26 (1927), p. 1, where other important references are given, and D. Hilbert, *Über das Unendliche*, Mathematische Annalen, vol. 95 (1925), p. 161; Hilbert, *Über das Unendliche*, Jahresbericht der Vereinigung, vol. 36 (1927), p. 201. Of interest for their general bearing upon these questions are also the following papers: Cipolla, *Sue fondamenti logici della matematica secondo le recenti vedute di Hilbert*, Annali di Mathematica, (4), vol. 1 (1923), p. 19; B. Levi, *Sui procedimenti transfiniti*, Mathematische Annalen, vol. 90 (1923), p. 164; A. Müller, *Über Zahlen und Zeichen*, Mathematische Annalen, vol. 90 (1923), p. 153; M. Pasch, *Betrachtungen zur Begründung der Mathematik*, Mathematische Zeitschrift, vol. 20 (1924), p. 231; J. v. Neumann, *Eine Axiomatisierung der Mengenlehre*, Journal für Mathematik, vol. 154 (1925), p. 219.

Middle. In the forms in which they usually appear in text-books on logic, they state respectively that (1) "$A$ is $A$"; (2) "$A$ is $B$" and "$A$ is not $B$" cannot hold simultaneously; and (3) of the two propositions "$A$ is $B$" and "$A$ is not $B$," one must always hold, whatever $A$ and $B$ may be. Brouwer's departure consists in this, that he does not accept the L.E.M. as a valid basis for all mathematical reasoning. I shall not here enter upon the reasons for his lack of confidence in the L.E.M.; but I shall refer to the logical basis which leaves it out of account as non-Aristotelian. This designation, apart from obvious reasons, is selected in order to point to an analogy with non-euclidean geometry, to which, although it is obvious, attention does not seem to have been called.* At the same time I must point out one important difference between the non-euclidean geometries and this non-Aristotelian logic. For, whereas in the former, Euclid's parallel postulate is replaced by another postulate, this logic merely omits the L.E.M. from the fundamental canons of logic. And it is in this respect that Barzin and Errera do not, in my judgment, interpret Brouwer's position correctly. For, in order to prove that this position must inevitably lead to a contradiction, they replace the L.E.M. by a new canon, namely that of the excluded fourth. Now I would be the last one to deny them the right to do this; but in so doing they create a non-Aristotelian logic different from that of Brouwer, so that their conclusion can not, in itself, be taken as a refutation of Brouwer's position. It must indeed be clear that any attempt to show that Brouwer's position which uses only two of the Aristotelian canons, leads to a contradiction, would, if successful, not leave any hope for the consistency of the classical logic. It is not likely therefore that the position can be refuted by this method.

For the purpose of my discussion, I shall make an arbitrary

* When we take cognizance of the controversial literature which has grown up in this field, we get a vivid realization of the reasons which may have led Gauss to withhold his discovery of non-euclidean geometry from publication.

division of the content of the mathematical sciences. We shall distinguish between pure mathematics (the word is to be understood as a plural here) and the application of mathematics, the latter to be taken in a sense rather different from the usual one, in which the line of demarcation is rarely definitely fixed. As a pure mathematic I designate any structure which proceeds from a set of primitive ideas, arbitrarily selected and named, and a set of primitive propositions concerning these ideas (assumptions or postulates), arbitrarily laid down, and which develops combinations of these ideas and propositions without the intervention of extraneous concepts or assumptions; the method of development is thus in an essential sense a constructive process. It is of course clear that the initiative for the making of these combinations, particularly of significant and fruitful combinations, does not lie in the assumptions and primitive ideas, but must arise out of a "capacity of the human mind," without which we could not get a step away from the axiomatic basis. The recognition of the very significant intervention of the human mind is an important element in our point of view. How it operates and why is not a part of our subject, in spite of its title; this must remain a subject for psychological investigation. An application of a mathematic is made whenever a one-to-one correspondence is set up between the primitive ideas of the mathematic and entities which have in some way or other secured objective existence, in such manner that the primitive propositions receive objective verification when these entities are put in place of the primitive ideas entering into the primitive propositions. It is at this point that a peculiarity of my division of the mathematical content into pure mathematics and applications of mathematics makes itself felt. For the entities, through the introduction of which an application of one mathematic is made, may be the primitive ideas of another mathematic, and the objective existence which is attributed to them is of a different order from that which is attached to chairs and tables. If we call such objective

existence as is enjoyed by entities which, to speak without making an attempt at great refinement, can be apprehended by the senses, as being of order zero, we may say that primitive ideas which enter into a mathematic of which an application can be made through the introduction of such entities of zero order, have themselves objective existence of order one. Proceeding in this way we introduce objective existence of increasing orders; the higher the order of objective existence of an entity, the farther it is removed from what is ordinarily designated by the term. It is of course clear furthe more, that the designation of an order of objective existence attached to an entity is not a unique process, but depends upon the constructive procedure by means of which it has been reached. It is of secondary importance for our present purpose whether minimum order of objective existence greater than or equal to 1 can be attached in a unique way to every entity with which mathematicians deal, although this appears to be an interesting question for further study. In the same manner, we can assign an order to propositions, calling facts which are directly verifiable in the world of the senses, propositions of order zero; propositions which are verified by means of them, propositions of order one, and so forth. If, for instance, we set up a mathematic consisting of primitive propositions stated in terms of elements $a$ and operations $(+)$ and $(\times)$; and if these primitive propositions are verified by facts that are directly observable when we substitute, say, apples or pebbles for the elements $a$ and some concrete operations for $(+)$ and $(\times)$, then these elements $a$ and the operations $(+)$ and $(\times)$ obtain objective existence of order 1, and the primitive propositions, together with the consequences drawn from them, become propositions of order 1. If now we set up a new mathematic in terms of elements $b$ and operations $((+))$ and $((\times))$, whose primitive propositions reduce to propositions of order 1, when, for example, pairs of elements $a$ and the operations $(+)$ and $(\times)$ are put in place of the elements $b$ and the operations $((+))$ and $((\times))$, re-

spectively, then the elements $b$ and the operations $((+))$ and $((\times))$ obtain objective existence of order 2; and so forth. The situation I am describing is somewhat obscured in many of the examples of postulational theories which are known in the literature; for in those cases, we usually set up postulates for a particular theory, that is to say, we construct a set of postulates that is to be applicable to a mathematic which has secured citizenship not by coming into the country through an immigration office, but by means of settlement before immigration laws were put into effect, or perhaps by having been smuggled in. For instance, when a set of postulates for positive integers in terms of elements $a$ and operators $(+)$ and $(\times)$ is applied, we usually make appeal not to entities and propositions for which an order of objective existence has been definitely established, but to concepts with which we are well acquainted, to neighbors with whom we are on intimate terms, even though we have never seen their naturalization certificate, indeed though we have never inquired whether they had one. But if we were to proceed on a strictly legal basis the testimony of such citizens could not be admitted; we would have to depend on original settlers or on such as possessed definitely traceable citizenship. That is to say, application of a set of postulates would then be made only through verification in terms of entities for which an order of objective existence can be established, and, perhaps preferably, in terms of entities of objective existence of order zero.

This digression on the peculiar character of the division which we have introduced into the content of mathematics seemed desirable in order to avoid misunderstanding; for our main purpose, the only point of importance is that application of a mathematic must ultimately depend upon the intervention as sponsors for our primitive ideas of entities which have, speaking roughly, the objective existence of chairs and tables, that is, what we have called objective existence of order zero, and that the verification of primitive propositions must ultimately depend upon propositions of

order zero. We can sum up this conclusion in the statement that the applicability of a mathematic must ultimately depend upon the possibility of setting it into one-to-one correspondence with a part of the content of human experience.

Returning now to the consideration of a pure mathematic, it must be clear that there is no immediate meaning in the question whether it is true.* The only sense in which such a question can be made significant is by interpreting it as an inquiry as to the applicability of the mathematic, in the sense in which this term has just been explained. Each one of us can build up a mathematic, if only we are sufficiently inventive to originate a set of primitive ideas and primitive propositions. Whether these mathematics will be fruitful of results, whether they will command the abiding interest not only of their inventors but also of others, will depend upon the answers to two questions, namely whether they stimulate minds to exercise in a lasting manner that capacity for constructive combination to which we have made reference before, and whether they are applicable to significant parts of the content of human experience. I am very doubtful in my own mind whether a mathematic can have the former of these two characteristics without having the latter; but this question I must also leave aside. Certainly it seems to me so that if a mathematic has neither of the two, it will be completely sterile and will deservedly die an early death. If it have the latter property as well as the former, it will surely arouse the interest of the worker in applied science, of the "practical" man; if the property of applicability is possessed in an easily recognizable way, if the

* It is perhaps not without interest to quote here the following passage from a definitely non-mathematical source: "What was truth, after all? It was a very sensible question of Pilate's. Perhaps there *was* no truth, nor falsehood either, in any actual set of words arranged in a certain order. Perhaps they were only a neutral surface over which truth or falsehood could be cast by different minds or tongues, as a blue or an amber light is projected by turns on a colourless piece of stage canvas." (C. E. Montague, *Right Off The Map*, p. 183.)

mathematic can, through its applications, penetrate deeply into human experience, it may appeal even to the man in the street. It is, however, on account of the former property primarily that a mathematic will command the attention of mathematicians as such, because it offers them an opportunity for the investigative ability which is peculiarly theirs, namely the initiating of combinations of relatively simple elements which are rich in possibilities for further development. Of especial significance for the mathematician will be the comparison of different mathematics which are related to each other. Such relations may arise through the partial identity of their fundamental content. If two mathematics have their primitive ideas and some of their primitive propositions in common (or if there is a simple isomorphism between their primitive ideas and some of their primitive propositions), the development of either illuminates in a striking way the structure of the other. Indeed it seems to me so, that full insight in a mathematic can be gained only by comparing it with several of its related mathematics. Every one will realize that this statement is little more than a generalization by extension of the marvelous enrichment of geometrical concepts to which the non-euclidean geometries have given rise, and of the enlargement of our ideas concerning geometry and physics which the theory of relativity has brought about.

We are now in a position to view in what seems to me to be its true light the second of the circumstances mentioned at the beginning of this paper. Whether or not one agrees with the opinion according to which logic is a part of mathematics, there will be little opposition to the thesis that logic plays a fundamental role in every mathematic. And this is so because the development of a mathematic, the constructive process which we have recognized as the driving force of this development, proceeds according to the laws of logic. Indeed in every mathematic, or at least in nearly every mathematic, that has been developed thus far, it is understood, either explicitly or tacitly, that the develop-

ment should remain in accord with the canons of Aristotelian logic. But, whether Aristotelian or non-Aristotelian, it must be clear that a logic must be the inherent part of every mathematic, that the primitive ideas and primitive propositions of some logic are a substructure of every mathematic. It follows therefore that every mathematic may be looked upon as involving an application of some logic, because the canons and primitive ideas of a logic must be considered as having obtained in it some sort of objective existence; they have to be appealed to if the mathematic is ever to develop beyond the embryonic stage. This carries with it, of course, that at least in so far as its structure is concerned the logic must, in some way, be looked upon as a mathematic; but for its development we can evidently not rely upon another logic, for this would clearly involve us in an infinite regress. Indeed the logic must be thought of as a part of what I have referred to above* as "the capacity of the human mind"; and the development of the logic has to proceed by the use of the residue of this capacity, left after the separating off of the logic. This residue we shall call the "bare capacity of the human mind."†

It has been recognized that the logic which underlies a mathematic has a very important bearing upon its development; and for this reason, writers, such as Frege, Peano, Whitehead and Russell have set up a pure logic as the preliminary for any mathematic. But this has been done only for the logic of classes, propositions and relations, and has not affected the fundamental canons. The significance of the departure which Brouwer has made lies in this, that he has brought the canons of logic explicitly into the foreground. By suggesting the possibility of modifying these canons as the fundamental basis even of the logic of mathematics, he

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* See page 442.

† The significance of the explicit formulation of the Aristotelian canons lies in the fact that they constituted, as far as I am aware, the first attempt at an analysis of the "capacity of the human mind." Let us not fall into the error of thinking that they are a complete formulation of that capacity

has called attention to the fact that they also are a part of this basis; that what we have called the "capacity of the human mind" need not include all the Aristotelian canons. Thus he has freed the mind from the compulsory use of the Aristotelian base. The significant question to ask concerning this new logic which Brouwer builds on is not whether it is true, but whether it is applicable; and this question, in the light of our earlier discussion, means whether it is possible to set its primitive ideas and propositions into one-to-one correspondence with entities that have objective existence of order zero in human experience, that is to say in short, whether it is possible for the human mind to operate effectively with it. If it is, then the developments of this logic will aid greatly in giving insight into the capacities and possibilities of the mind, just as the comparison of two closely related postulate systems for one field of mathematics enrich our knowledge of that field. And there are indications that it is indeed possible to do so in the constructive work that has already been done by Brouwer,* Heyting† and others. The developments of a mathematic by means of this non-Aristotelian logic (usually called intuitionist mathematics) should then be looked upon in the same way as we look upon non-euclidean geometry, as an enrichment of our understanding of the way in which the human mind operates. For myself, I am very doubtful whether I could trust myself not to use the L.E.M.; but I am ready to admit that any one who can, may do important work in that way. I would go farther and open the way for using various types of logic. Each one of us can choose to work with one logic to-day and with another to-morrow, just as one's work in non-euclidean geometry does not exclude him forever from working in euclidean geometry, or as dealing with non-commutative algebras one year need not incapacitate one for occupying

* See the list of references given by Barzin and Errera in the article quoted on p. 440.

† See *Zur intuitionistischen Axiomatik der projektiven Geometrie*, Mathematische Annalen, vol. 98 (1927), p. 491.

himself with commutative algebras the next. Whether or not his work will be fruitful is difficult to decide in advance.

If this were all there is to the position which I am taking, it would probably not be objected to seriously. For it gives greater liberty, leaving open not merely the primitive ideas and primitive propositions of a mathematic to choice, but also the rules which are to govern its development. But there is another aspect to this story, which we must now consider briefly. For the freedom which we gain is obtained at the sacrifice of a certain permanence of existence which mathematicians have usually wanted to attribute to their science, an existence in essence which is independent of any mind, and independent of time, which requires discovery to have it brought down to the realm of human apprehension. Only recently I came across the following passage in a letter from Hermite to Konigsberger, in which this idea finds expression. He wrote: "I add that these notions of analysis have their existence apart from us, that they constitute a whole of which only a part is revealed to us, incontestably although mysteriously associated with that other totality of things which we perceive by way of the senses."* To this belief, which is probably shared by most mathematicians to-day, the position which I have developed leaves little support. For if the entire mathematical structure, not the basis only, but also the guiding principles for its development, is at the choice of the individual, we have to admit that a mathematic exists only in the minds of the individual, and that without the activity of the human mind there would be no mathematics. This carries with it furthermore, that a mathematic exists only in so far as it has been developed, that is, is invented rather than discovered; that not only with regard to primitive ideas and primitive propositions, but also with regard to the logical processes, the question of truth is irrelevant; that mathematical entities may exist

* See W. Birkemeier, *Über den Bildungswert der Mathematik*, Teubner, 1923, p. 26.

to-morrow which do not exist to-day, that their existence depends upon the construction of a process, which can call them into being. It seems to me that Brouwer's position inescapably commits us to this conclusion; it clarifies points in his work, which otherwise present difficulties which it seems impossible to overcome. By disregarding this consequence of his position in their critique of his work,* Barzin and Errera seem to me to misconceive a fundamental aspect of it. In one of his most interesting papers,† Brouwer introduces an integer $k$ which is defined as the order number of that place in the decimal development of $\pi$, at which there appears for the first time a zero, immediately followed by the sequence of digits 1, 2, $\cdots$, 9. When we use the non-Aristotelian logic which omits the L.E.M., we say that we do not know whether this $k$ exists, nor whether we shall ever be able to answer the question whether it exists, and that we are consequently able to construct a real number of which we do not know whether it is less than, equal to or greater than 0. If, on the other hand, we use the Aristotelian logic, we can say, using the L.E.M. that either this number $k$ exists or else it does not exist, independently of what we know about it, and that the real number determined by the use of this $k$ is either less than, equal to or greater than 0, whether or not we know which of these alternatives holds. That is to say we hold that the mathematical fact has existence, independently of whether we have discovered it.

There is therefore a fundamental difference between the characters of mathematical existence which the two points of view involve. But is this cause for alarm? I think not; in diversity there may lie strength. Indeed the new point of view adds to our insight. For when we learn that, without the use of the L.E.M. the system of real numbers is not ordered, we have increased our knowledge, just as when

* See Barzin and Errera, loc. cit., p. 59.

† See Brouwer, *Über die Bedeutung des principium tertii exclusi in der Mathematik, besonders in der Funktionentheorie*, Journal für Mathematik, vol. 154 (1925), p. 1.

we learn that in matrix algebra the product of two factors can vanish, while neither factor vanishes. In which of the two ways we proceed is a matter of choice to be determined by each individual, not for all time necessarily, but from time to time, according to his tastes and prejudices.

We come finally to the question of the bearing of this point of view on the validity of mathematical conclusions. It would seem as if the freedom which is left to the individual to build his mathematic as he chooses would not leave much basis on which to rest a compelling faith in his conclusions. Indeed, as we have seen before, the question as to the truth of these conclusions would not have any meaning. But the non-mathematician is not interested in a mathematic, but rather in the applications of a mathematic, indeed in the application which leads back from the mathematic to the content of human experience. Now we have seen that if a mathematic is to survive beyond the embryonic stage, it must be capable of development and of application. Thus a natural selection takes place which secures for a surviving mathematic those qualities which link it up with the realm of human experience. The canons of logic by which the development takes place must have their roots in the experiences of the mind; the primitive ideas and the primitive propositions must have their ultimate connections with the objective experience of the race, they must be, as it were, the result of a process of distillation, which continues throughout the history of the race; they must embody the essence of this experience, obtained through successive abstractions from its significant elements. In this way the application of a mathematic brings forth results which are relevant in human experience and which are in harmony with its fundamental characteristics. And it is this harmony of the applications of a mathematic with fundamental aspects of his experience, which constitutes their truth for the man in the street, for the man not directly concerned with mathematics; from it he derives his faith in the validity of mathematical conclusions.

We may sum up our conclusions as follows. A mathematic may be established through the free choice of a logic, and of primitive ideas and primitive propositions; if this choice is guided by wisdom, the mathematic will be capable of development and application. The application will then, sometimes directly, but more frequently through a chain of intermediate stages, have significant bearing upon the content of human experience and furnish results which may be called true. Their truth then gives, retroactively, a sound basis for belief in the validity of the conclusions of mathematics.

SWARTHMORE COLLEGE

# RECENT DEVELOPMENTS IN PROJECTIVE DIFFERENTIAL GEOMETRY*

BY E. B. STOUFFER AND E. P. LANE

## A. Invariants and Covariants, Canonical Forms, Canonical Expansions

1. *Introduction.* In a long series of papers appearing first in 1901 and continuing for more than twenty years, Wilczynski made a study of the projective differential properties of many geometrical figures, including plane and space curves, ruled and curved surfaces, and linear congruences. While Wilczynski was not the first to discover projective differential properties of figures, he was the first to proceed in a *systematic* manner in finding these properties. His general scheme in studying a given figure is to set up a system of one or more linear homogeneous differential equations such that the fundamental sets of solutions of the system determine the figure uniquely except for projective transformations. The independent and the dependent variables which appear in the differential equations and also in the parametric equations of the figure can be subjected to certain transformations which do not disturb the figure or change the form of the system of differential equations, but will in general change the coefficients of the differential equations. A function of the new coefficients and their derivatives and of the new dependent variables and their derivatives which is equal, except possibly for a factor, to the same function of the original coefficients and variables is called a *covariant.* A covariant which does not contain the dependent variables or their derivatives is called an *invariant.*

Wilczynski uses a complete and independent system of invariants and covariants as a foundation for each of his

*Two addresses presented at the request of the program committee at the Western meeting of the Society, April 6, 1928. Part A was given by Professor Stouffer, and Part B by Professor Lane.

geometrical studies. His general method of calculating the invariants and covariants is by means of the Lie theory of continuous groups, a process which is frequently very laborious, although the final results are usually quite elegant. Since the method is purely analytical, there is no assurance that the invariants and covariants will appear in a form which will readily show their geometric significance.

Fubini and his school began about 1913 to study projective differential properties by means of differential forms. However, before many geometrical results can be obtained by this method it is necessary in each case to pass from the forms to a system of differential equations of the type used by Wilczynski. Fubini confines his attention almost wholly to systems which have been made canonical, that is, systems in which the coefficients have been specialized by means of the permissible transformations in such a way as to make the fundamental covariants take very simple forms.

One purpose of this paper is to derive several canonical forms for defining systems of differential equations and to show how each properly chosen canonical form leads in a very simple manner to a complete system of invariants and covariants of the Wilczynski type. A second purpose is to show how such canonical forms make possible direct derivation of canonical expansions for the equations of the figures.

In a paper[1] published in 1915 Green showed how the invariants and covariants associated with a curved surface referred to general parametric curves could be obtained from those associated with the surface referred to the asymptotic curves as parametric curves. The principle which Green used in that particular problem is of remarkably wide application. We shall first apply it to the very simple case of a plane curve.

2. *Curves.* The differential equation associated with a plane curve $C$ is of the form[2]

$$y''' + 3p_1y'' + 3p_2y' + p_3y = 0, \tag{1}$$

[2] Numerical references are made to the bibliography at the end.

where differentiation is with respect to the independent variable $x$ and where $p_i$ are functions* of $x$. The permissible transformations in this case have the form

$$\bar{x} = \phi(x), \tag{2}$$

$$y = \lambda \bar{y}, \tag{3}$$

where $\phi$ and $\lambda$ are arbitrary functions of the independent variable.

The transformation (2) changes (1) into a new equation (1a) with coefficients $\bar{p}_i$ given by

$$\begin{cases} \bar{p}_1 = \dfrac{1}{\phi'}\left(p_1 + \dfrac{\phi''}{\phi'}\right), \\ \bar{p}_2 = \dfrac{1}{(\phi')^2}\left(p_2 + p_1\dfrac{\phi''}{\phi'} + \dfrac{\phi'''}{3\phi'}\right), \quad \bar{p}_3 = \dfrac{1}{(\phi')^3}p_3, \end{cases} \tag{4}$$

and (3) in turn changes (1a) into another equation (1b) in $\bar{y}$ with coefficients $\overline{P}_i$ given by†

$$\begin{cases} \overline{P}_1 = \bar{p}_1 + \dfrac{\bar{\lambda}'}{\bar{\lambda}}, \\ \overline{P}_2 = \bar{p}_2 + 2\bar{p}_1\cdot\dfrac{\bar{\lambda}'}{\bar{\lambda}} + \dfrac{\bar{\lambda}''}{\bar{\lambda}}, \\ \overline{P}_3 = \bar{p}_3 + 3\bar{p}_2\dfrac{\bar{\lambda}'}{\bar{\lambda}} + 3\bar{p}_1\dfrac{\bar{\lambda}''}{\bar{\lambda}} + \dfrac{\bar{\lambda}'''}{\bar{\lambda}}\cdot \end{cases} \tag{5}$$

It is easy to determine from (4) and (5) an invariant (2), that is, a function $\theta_3$ of $p_i$ which is related to the same function $\bar{\theta}_3$ of $\overline{P}_i$ by the equation $\bar{\theta}_3 = \theta_3/(\phi')^3$.

If we choose $\phi'$ to satisfy the equation‡ $(\phi')^3 = \theta_3$ so that $\bar{\theta}_3 = 1$ and then choose $\bar{\lambda}$ to satisfy the equation $\bar{\lambda}'/\bar{\lambda} = -\bar{p}_1$

*Here as everywhere else in this paper we shall assume that all functions which appear are capable of differentiation to as high an order as neccessary.

†In equations (5), $\bar{\lambda}$ is a function of $\bar{x}$ and differentiation is with respect to $\bar{x}$.

‡It is assumed that $C$ is not a conic, since in that case $\theta_3 = 0$.

so that $\overline{P}_1 = 0$, we have a canonical form of the desired type for our differential equation. This is exactly the canonical form obtained by Sannia[3] who introduced the differential form $a_1 dx$ and then chose $a_1 = \theta_3^{1/3}$.

Since the conditions $\bar{\theta}_3 = 1$, $\overline{P}_1 = 0$ are maintained by the transformations (2) and (3) only if $\phi'' = \lambda' = 0$, the coefficients $\overline{P}_i$ of our canonical form and their derivatives are each determined except for possible factors. A simple process of reasoning shows that they are the canonical forms of invariants whose general forms in terms of the coefficients of (1) can be obtained from (5) by direct substitutions from (4) with the now known values of $\phi'$ and $\lambda'$ inserted. The same reasoning shows that $\bar{y}$ and its derivatives are the canonical forms of covariants whose general forms follow by mere substitution. The completeness of the system of invariants and covariants thus obtained is evident and the determination of the particular invariants and covariants which are independent is not at all difficult.

The system of invariants and covariants thus obtained is, of course, equivalent to the system derived by Wilczynski and is, in fact, not greatly different from it. The great value of the method outlined above lies in its simplicity and in its complete avoidance of the complicated Lie theory.

A canonical form of the above type for the differential equation of a plane curve leads by a process involving only direct substitutions to an equation for the curve in the form of an expansion of one non-homogeneous coordinate in terms of the other.[4] This method of obtaining a canonical expansion has the advantage that the vertices of the triangle of reference are exactly the points determined by the three covariants $\bar{y}$, $\bar{y}'$, $\bar{y}''$ and are, as a consequence, easily characterized geometrically. Since a canonical expansion is the most natural and simple tool for obtaining results in this geometry, the ability to derive the expansion in a direct analytic manner is of great advantage.

The general scheme applied above to plane curve theory can be applied to space curves.[4] However, it is desirable

in this case to use the canonical expansion in order to determine the most satisfactory canonical form for the fundamental differential equations, even though this canonical form is actually used to determine the canonical expansion. We shall illustrate this principle in some detail by means of ruled surface theory. In the study of ruled surfaces it has been long and difficult work to determine the invariants and covariants[2] and a canonical expansion[5] by previous methods.

3. *Ruled Surfaces.* Let us denote by $P$ the point on a ruled surface $R$ in the neighborhood of which we wish to study the surface. If we choose as one directrix curve $C_y$ the asymptotic curve through $P$ and for the other directrix curve $C_z$ any curve on the surface, it is well known[2] that the fundamental system of differential equations has the general form

$$(6)\qquad \begin{aligned} &y'' + 2p_{11}y' + \qquad\quad q_{11}y + q_{12}z = 0,\\ &z'' + 2p_{21}y' + 2p_{22}z' + q_{21}y + q_{22}z = 0, \end{aligned}$$

where $p_{ij}$ and $q_{ij}$ are functions of the independent variable $x$. The permissible transformations are

$$(7)\qquad \bar{x} = \phi(x),$$

for the independent variable and

$$(8)\qquad y = \alpha\bar{y}, \quad z = \gamma\bar{y} + \delta\bar{z}, \quad \alpha\delta \neq 0,$$

for the dependent variables, where $\phi$, $\alpha$, $\gamma$, $\delta$ are arbitrary functions of the independent variable.

Let us now assume that (6) has been transformed by (7) and (8) into a canonical form with the dependent variables and their first derivatives uniquely determined except for possible factors. What the canonical form is we do not know as yet. However, it is easy to determine that the associated canonical expansion for the equation of the surface in the neighborhood of the point $P$ is

$$(9)\qquad \zeta = \xi\eta + \frac{1}{3}\bar{Q}_{12}\xi^3 + \frac{1}{12}\bar{V}_{12}\xi^4 - \frac{1}{3}(\bar{U}_{11} - \bar{U}_{22})\xi^3\eta + \cdots,$$

where $\overline{V}_{12}$ and $\overline{U}_{11}-\overline{U}_{22}$ are simple functions of the coefficients $\overline{P}_{ij}$ and $\overline{Q}_{ij}$ of the canonical form and their derivatives. Equation (9) makes evident the desirable conditions to be imposed in order to produce a canonical form and at the same time simplify the expansion. By means of (7) and (8) we can make*

$$\overline{V}_{12} = \overline{U}_{11} - \overline{U}_{22} = \overline{P}_{11} = \overline{P}_{22} = 0, \ \overline{Q}_{12} = 1.$$

Since these conditions are maintained by (7) and (8) only if $\phi''=\alpha'=\delta'=\gamma=0$, we obtain for (6) a canonical form

$$(10) \qquad \begin{cases} \bar{y}'' + \qquad\qquad \overline{Q}_{11}\bar{y} + \bar{z} = 0, \\ \bar{z}'' + 2\overline{P}_{21}\bar{y}' + \overline{Q}_{21}\bar{y} + \overline{Q}_{11}\bar{z} = 0 \end{cases}$$

in which the coefficients and their derivatives are invariants and the dependent variables and their derivatives are covariants, all in their canonical forms. The general form of this complete system of invariants and covariants in terms of the coefficients and variables of (6) can be obtained by direct substitutions just as the case of plane curves.

The expansion (9) is thus reduced to the simple form $\zeta=\xi\eta+\xi^3/3+$ terms of at least the fifth degree. The vertices of the tetrahedron of reference are given at once by the four simplest covariants. Their geometrical significance is easily determined. In fact, one vertex is the point $P$, a second is the point $\overline{P}_z$, the harmonic conjugate of $P$ with respect to the flecnode points on the generator through $P$, a third is the point $\overline{P}_\rho$ on the tangent to $C_y$ at $P$ which is the pole of the generator with respect to the osculating conic of $C_y$ at $P$, and the fourth is the intersection of the asymptotic tangent at $\overline{P}_z$ and the line of the osculating hyperboloid which passes through $\overline{P}_\rho$.

The geometrical significance of the vanishing of the invariants, which in their canonical forms are the coefficients of (10), can be read directly from the differential equation.

*These results are not valid if $P$ is a point on a flecnode curve.

In fact, if $\overline{P}_{21}=0$, $\overline{C}_z$ is an asymptotic curve, if $\overline{Q}_{21}=0$ the flecnode curves coincide and if $\overline{Q}_{11}=0$ the tangent at $\overline{P}_\rho$ to $\overline{C}_\rho$ passes through $\overline{P}_z$.

Čech[6] has applied Fubini's methods to ruled surfaces by starting with a specialized case of the differential forms for curved surfaces. He derives from these forms a pair of second order differential equations which he makes canonical by putting a certain invariant $\theta_4$ of Wilczynski equal to unity.

4. *Curved Surfaces.* In applying the above methods to curved surfaces we start with the fundamental equations[7]

$$\begin{aligned} &y_{uu}+2a\,y_u+2by_v+cy=0,\\ &y_{vv}+2a'y_u+2b'y_v+c'y=0, \end{aligned} \tag{11}$$

with the asymptotic curves parametric. The permissible transformations are

$$\bar{u}=\phi(u),\qquad \bar{v}=\psi(v), \tag{12}$$

for the independent variables, and

$$y=\lambda(u,v)\bar{y}, \tag{13}$$

for the dependent variable.

Let us assume that (11) has been transformed by (12) and (13) into a system with coefficients $\overline{A}$, $\overline{B}$, $\overline{C}$, $\overline{A}'$, $\overline{B}'$, $\overline{C}'$, whose dependent variable $\bar{y}$ and its derivatives $\bar{y}_{\bar{u}}$, $\bar{y}_{\bar{v}}$, $\bar{y}_{\bar{u}\bar{v}}$ are all completely determined except for possible factors. It is easily found that an associated canonical expansion for the equation of the surface $S$ in the neighborhood of a point $\overline{P}_y$ given by $\bar{u}=\bar{u}_0$, $\bar{v}=\bar{v}_0$, is

$$\left\{\begin{aligned} \zeta=\;&\xi\eta+\tfrac{2}{3}\overline{B}\xi^3+\tfrac{2}{3}\overline{A}'\eta^3+\tfrac{2}{3}(\overline{B}_{\bar{v}}-2\overline{B}\overline{B}')\xi^3\eta\\ &+\tfrac{2}{3}(\overline{A}'_{\bar{u}}-2\overline{A}\overline{A}')\xi\eta^3+\tfrac{1}{6}(\overline{B}_{\bar{u}}+4\overline{A}\overline{B})\xi^4\\ &+\tfrac{1}{6}(\overline{A}'_{\bar{v}}+4\overline{A}'\overline{B}')\eta^4+\tfrac{1}{2}(\overline{A}_{\bar{v}}+\overline{B}_{\bar{u}})\xi^2\eta^2+\cdots. \end{aligned}\right. \tag{14}$$

The fact that $\phi(u)$ and $\psi(v)$ in the transformation (12) are each functions of only a single variable limits the conditions which may be imposed by this transformation. However, we can, for instance, impose by (12) and (13) the conditions

$$\overline{A}'_{\bar{u}}(\bar{u},\bar{v}_0) = \overline{B}_{\bar{v}}(\bar{u}_0,\bar{v}) = 0, \tag{15}$$

$$\overline{A} = \overline{B}' = 0, \tag{16}$$

conditions which are maintained by (12) and (13) only if $\phi_{uu}=\psi_{vv}=\lambda_u=\lambda_v=0$. Since the coefficients in (14) are functions of $\bar{u}_0$ and $\bar{v}_0$ we are thus led to a canonical expansion for the equations of the surface. However, we do not obtain in place of (11) a canonical form in the true sense. Nevertheless, by the same kind of simple substitutions as above, the coefficients and variables of the reduced form and their derivatives produce a complete system of invariants and covariants in terms of the coefficients and variables of (11). The expansion (14) becomes under the conditions (15) and (16) essentially that obtained by Wilczynski[8] and also by Green[9], with the vertices of the tetrahedron of reference determined by the simplest covariants and located at the intersection of Wilczynski's directrices of the first and second kind with his canonical quadric[8].

In place of (15) we may impose the more general conditions

$$l\frac{\overline{A}'_{\bar{u}}(\bar{u},\bar{v}_0)}{\overline{A}'(\bar{u},\bar{v}_0)} + m\frac{\overline{B}_{\bar{u}}(\bar{u},\bar{v}_0)}{\overline{B}(\bar{u},\bar{v}_0)} = l\frac{\overline{B}_{\bar{v}}(\bar{u}_0\ \bar{v})}{\overline{B}(\bar{u}_0,\bar{v})} + m\frac{\overline{A}'_{\bar{v}}(\bar{u}_0,\bar{v})}{\overline{A}'(\bar{u}_0,\bar{v})} = 0 \tag{17}$$

where $l$ and $m$ are constants. In each case we obtain a canonical expansion with two points $\overline{P}_z$ and $\overline{P}_\rho$, one on each asymptotic tangent through $P_y$, as two vertices of the tetrahedron of reference. The fourth vertex $\overline{P}_\sigma$ is the intersection of the canonical quadric of Wilczynski with the polar reciprocal of the line $\overline{P}_z\overline{P}_\rho$ with respect to the quadric of Lie. For all values of $l$ and $m$ the lines joining corresponding points $\overline{P}_z$ and $\overline{P}_\rho$ pass through a point, the canonical point, and the corresponding lines joining $P_y$ and $\overline{P}_\sigma$ all lie in a plane, the canonical plane.

If $m=0$ we have the case mentioned above. If $l=0$ the vertices are on the canonical edges of Green[9]. If $l/m=1$ the line joining $P_y$ and $\overline{P}_\sigma$ is the pseudo-normal of Green[9], the projective normal of Fubini[10]. Other values of the

ratio $l/m$ give other covariant lines. In each case the coefficients and variables of the reduced form produce a corresponding system of invariants and covariants which can be expressed in terms of the coefficients and variables of (11) by direct substitutions.

If we apply the transformations (13) alone, we find that it is possible to impose the conditions[11]

$$2\overline{A} + (\log \overline{A}'\overline{B})_u = 2\overline{B}' + (\log \overline{A}'\overline{B})_v = 0. \tag{18}$$

These conditions are not disturbed by the general transformation (12). If we write $\theta = (1/2) \log \overline{A}'\overline{B}$, equations (11) become under the conditions (18)

$$\begin{aligned} &\bar{y}_{uu} - 2\theta_u \bar{y}_u + 2\overline{B}\bar{y}_v + \overline{C}\bar{y} = 0, \\ &\bar{y}_{vv} + 2\overline{A}'\bar{y}_u - 2\theta_v \bar{y}_v + \overline{C}'\bar{y} = 0, \end{aligned} \tag{19}$$

which is the canonical form of Fubini.

If $\overline{A}$ and $\overline{B}'$ are replaced by $-\theta_u$ and $-\theta_v$, respectively, in accordance with (18), the expansion (14) becomes essentially the canonical expansion of Fubini[12]. The vertices of the tetrahedron of reference for this expansion are the same as for the case $l/m = 1$ above, except that the fourth vertex is the intersection of the projective normal with Fubini's canonical quadric[20]. Fubini[13] has derived the same expansion without the use of a fundamental system of differential equations. He simply assumes an expansion of one non-homogeneous coordinate in terms of the other two and then chooses his tetrahedron of reference properly.

Recently Lane[14] has obtained all the above mentioned canonical expansions for the equation of a curved surface, starting from the equivalent of equations (18). The same author[39] has also derived a significant canonical form for the equations of a curved surface referred to a conjugate net as the parametric net.

The methods outlined above for obtaining complete systems of invariants and covariants and canonical expansions apply equally well in many other cases. However, their consideration must be left to other occasions.

B. GEOMETRY OF SURFACES

1. *Surfaces in Ordinary Space.* The foundations for a theory of the projective differential geometry of an analytic non-ruled surface in ordinary space were laid by Wilczynski in a series of five memoirs which were published in the Transactions of this Society from 1907 to 1909. As the theory grew and expanded, Green made notable contributions to it, especially in a memoir that appeared in the Transactions after his death in 1919. The work of Fubini on this subject dates from about 1913, and that of Bompiani from about 1923.

A considerable portion of the projective differential geometry of a surface as elaborated in the last two decades, together with some of the more or less incidental results found by various geometers before 1907 can be organized about three unifying ideas, namely, quadrics containing the asymptotic tangents, the canonical pencils of lines, and union curves with their generalizations. After establishing an analytic basis for our discussion we shall consider each of these topics in turn.

Let the projective homogeneous coordinates $x^{(1)}, \cdots, x^{(4)}$ of a point $P$ on a surface $S$ be functions of two independent variables $u, v$. If the asymptotic net is parametric and if the proportionality factor of the coordinates is suitably chosen, then the functions $x$ are solutions of a completely integrable system of differential equations of the form

$$(1) \quad x_{uu} = px + \theta_u x_u + \beta x_v, \quad x_{vv} = qx + \gamma x_u + \theta_v x_v, \qquad (\theta = \log \beta\gamma).$$

The coordinates of a point $N$ on $S$ near $P$ can be represented by Taylor's formula as power series in the increments $\Delta u$, $\Delta v$ corresponding to displacement from $P$ to $N$. Then by means of (1) and the equations obtained therefrom by differentiation it is possible to express each of these series uniquely in the form $x_1 x + x_2 x_u + x_3 x_v + x_4 x_{uv}$, where $x_1, \cdots, x_4$ are the following series which represent the local coordi-

nates of $N$ referred to the covariant tetrahedron $x$, $x_u$, $x_v$, $x_{uv}$, with suitably chosen unit point:

$$(2)\quad \begin{cases} x_1 = 1 + (p\Delta u^2 + q\Delta v^2)/2 + \cdots, \\ x_2 = \Delta u + (\theta_u \Delta u^2 + \gamma \Delta v^2)/2 + \cdots, \\ x_3 = \Delta v + (\beta \Delta u^2 + \theta_v \Delta v^2)/2 + \cdots, \\ x_4 = \Delta u \Delta v + (\beta \Delta u^3 + 3\theta_u \Delta u^2 \Delta v + 3\theta_v \Delta u \Delta v^2 + \gamma \Delta v^3)/6 + \cdots. \end{cases}$$

The equations of the $u$-tangent are $x_3 = x_4 = 0$, and those of the $v$-tangent are $x_2 = x_4 = 0$.

2. *Quadrics containing the Asymptotic Tangents.* The equation of any non-singular quadric surface $\phi$ containing the asymptotic tangents of $S$ at $P$ can be written [15] in the form

$$(3)\quad x_2 x_3 + x_4(K_1 x_1 + K_2 x_2 + K_3 x_3 + K_4 x_4) = 0 \quad (K_1 \neq 0).$$

If $K_1 = -1$, then $\phi$ has contact of the second order with $S$ at $P$ and cuts $S$ in a curve with a triple point at $P$. If the triple point tangents coincide, they coincide in one of *the three directions of Darboux*[16] for which

$$(4)\qquad \beta du^3 + \gamma dv^3 = 0.$$

If $K_1 = -1$ and $K_2 = K_3 = 0$, then $\phi$ is *a quadric of Darboux*, which has contact of the second order and has the tangents of Darboux for triple point tangents.

Among the quadrics of Darboux there are four that deserve mention. If $K_4 = -(\beta\gamma + \theta_{uv})/2$, then $\phi$ is *the quadric of Lie*[17], called by Wilczynski *the osculating quadric*, which is the limit of the quadric determined by three asymptotic tangents of one family constructed at points of a fixed curve of the other family as these points approach coincidence along the fixed asymptotic. If $K_4 = -\theta_{uv}/2$, then $\phi$ is *the canonical quadric of Wilczynski*[8]. Bompiani has recently[18] rediscovered this quadric, apparently without recognizing it, by means of the following considerations. If a curve $C$ on $S$ has an inflexion at $P$ then $C$ is tangent to an asymptotic at $P$. The limit of the quadric determined

by three asymptotic tangents of the other family constructed at points of $C$, as these points approach $P$ along $C$, is the canonical quadric of Wilczynski. If in this definition the curve $C$, instead of having an inflexion, has a stationary osculating plane at $P$, the resulting quadric is the quadric of Fubini[19] for which $K_4 = -(2\beta\gamma + 3\theta_{uv})/6$. Finally, if $K_4 = 0$, then $\phi$ is *the canonical quadric of Fubini*[20], which is the quadric of Darboux that passes through the covariant point $(0, 0, 0, 1)$.

We shall refer briefly to only two more species of quadrics of the general type (3). At a point $P$ of curve $C$ on $S$ there are two of Bompiani's *asymptotic osculating quadrics*[21]; each of these is the limit of the quadric determined by three asymptotic tangents of one family constructed at points of $C$, as these points independently approach $P$ along $C$. And *the quadric of Moutard*[22] of $S$ at $P$ in the direction of a tangent $t$ is the locus of the osculating conics at $P$ of the curves of intersection of $S$ and the planes of the pencil with $t$ as axis.

3. *The Canonical Pencils.* The canonical line pencils of $S$ at $P$ may be defined analytically as follows. The line $l_1$ joining $P$ to the point $(0, K\psi, K\phi, 1)$, where

$$\phi = \frac{\partial}{\partial u} \log \beta\gamma^2, \quad \psi = \frac{\partial}{\partial v} \log \beta^2\gamma, \qquad K = \text{const.}, \tag{5}$$

is a canonical line of the first kind. When $K$ varies, the locus of $l_1$ is *the first canonical pencil*, with center at $P$ and lying in *the canonical plane* $\phi x_2 - \psi x_3 = 0$. The polar line $l_2$ of $l_1$ with respect to the quadric of Lie is a canonical line of the second kind and joins the points $(K\phi, 1, 0, 0)$, $(K\psi, 0, 1, 0)$. When $K$ varies, the locus of $l_2$ is *the second canonical* pencil with center at *the canonical point* $(0, \psi, -\phi, 0)$ and lying in the tangent plane $x_4 = 0$.

Every canonical line is covariant to the surface, but negative rational values of $K$ give most of the lines that have appeared naturally in geometric investigations. For instance,

if $K=-1/2$, then $l_1$, $l_2$ are *the directrices*[8] of Wilczynski. These have several characteristic properties, but were discovered by Wilczynski as the directrices of the linear congruence of intersection of the osculating linear complexes of the two asymptotic curves at $P$. If $K=-1/4$, then $l_1$, $l_2$ are *the canonical edges*[9] of Green. Perhaps the most recently discovered characteristic property of these lines is due to B. Segre who shows[23] that all non-composite cubic surfaces having contact of the fourth order with $S$ at $P$ cut the tangent plane in the same cubic curve which has a double point at $P$ and has three inflexions lying on the second canonical edge of Green. If $K=-1/3$, then $l_1$ is *the axis* of Čech, called[24] by him *the line of Segre*, because the osculating planes of the three curves of Segre $\beta du^3-\gamma dv^3=0$ at $P$ intersect in this line. If $K=0$, then $l_1$ is *the projective normal*[10] of Fubini, which was discovered independently by Green and called[9] by him *the pseudo-normal*. Other canonical lines of interest are *the principal lines*[25] of Fubini, for which $K=-1/6$, $K=-1/12$; the lines[15] for which $K=-5/12$, $K=-3/4$; and[26] the lines[27] for which $K=-3/8$.

4. *Union Curves.* Let us consider a congruence $\Gamma$ of lines one of which, $l$, passes through each point $P$ of $S$ but does not lie in the tangent plane at $P$. *A union curve of* $\Gamma$ is defined[28] by Miss Sperry to be a curve on $S$ such that its osculating plane at each of its points contains the line of $\Gamma$ through the point. The differential equation of the union curves has the form

$$v''=A+Bv'+Cv'^2+Dv'^3 \qquad (v'=dv/du), \tag{6}$$

and the osculating planes at $P$ of all the union curves of $\Gamma$ through $P$ form a pencil with the line $l$ through $P$ as axis. For this reason Bompiani[29] calls the union curves of a congruence *an axial system*. In metric geometry the union curves of the congruence of normals are the geodesics.

The most general equation of the form (6) defines on $S$ a system of curves such that the osculating planes at a point

$P$ of all the curves of the system that pass through $P$ envelope a cone of the third class which touches the tangent plane of $S$ along the asymptotic tangents at $P$, and which has three cusp-planes that intersect in a line through $P$, called *the cusp-axis* of $P$ with respect to the given system of curves. We shall mention two examples of cusp-axes. The extremals of the integral

$$\int (\beta\gamma v')^{1/2} du$$

are given by Euler's equation,

$$v'' = \theta_u v'/\theta - \theta_v v'^2/\theta, \qquad (\theta = \log \beta\gamma), \tag{7}$$

and the corresponding cusp-axis is the projective normal. The curves of the Segre-Darboux pencil

$$v'' = \lambda_u v'/\lambda + \lambda_v v'^2/\lambda, \qquad (\lambda = (\beta/\gamma)^{1/3}). \tag{8}$$

have for cusp-axis the axis of Čech.

Bompiani defines[30] *a planar system* of curves as follows. Let us consider a congruence $\Gamma$ of quadrics one of which is associated with each point $P$ of $S$ and contains the asymptotic tangents through $P$ but does not have contact of order as high as the second with $S$ at $P$. Then *a planar curve of* $\Gamma$ is a curve on $S$ such that at each of its points its asymptotic osculating quadric of one family (which contains the asymptotic tangents through the point) intersects the associated quadric of $\Gamma$ in a residual pair of straight lines, instead of in a residual non-singular conic as would ordinarily be the case. The differential equation of a planar system is of the form (6). A given congruence $\Gamma$ determines two planar systems of curves because a curve has two families of asymptotic osculating quadrics. But if a planar system is defined by one family of asymptotic osculating quadrics and a congruence $\Gamma$, then a second congruence $\Gamma'$ can be determined so that it and the other family of asymptotic osculating quadrics define the same planar system. The reason for

the name *planar system* will appear when we discuss Bompiani's extension of this theory to hyperspace.

5. *Method of the Italian School.* Italian geometers of the school of Fubini define a configuration, except for a projective transformation, by means of a system of differential forms, and employ the absolute calculus of Ricci; whereas American geometers of the school of Wilczynski define a configuration projectively by means of a system of differential equations and make use of the Lie theory of continuous groups. Fubini's method[31] has been used by Sannia in studying plane and space curves; by Čech in his theory of ruled surfaces; and by Fubini in his investigations of rectilinear congruences and complexes. But the method finds perhaps its best exemplification in Fubini's theory of surfaces in ordinary space. We shall sketch this theory very briefly.

Let us consider a fundamental binary quadratic differential form

$$(9)\quad G = a_{ij}du_i du_j, \quad (a_{ij} = a_{ji}\,; A = \left| a_{ij} \right| \neq 0\,; i,j = 1,2),$$

which will be completely specified later, and let us also consider a surface $S$ whose parametric equations in projective homogeneous point coordinates are given by the equations $x^{(k)} = x^{(k)}(u, v)$ $(k = 1, \cdots, 4)$. Let us define two differential forms $F_2$, $\Phi_3$ by the equations

$$(10)\, F_2 = (x, x_1, x_2, d^2x) \left| A \right|^{-1/2}, \; \Phi_3 = (x, x_1, x_2, d^3x) \left| A \right|^{-1/2},$$

wherein numerical subscripts of $x$ indicate covariant differentiation with respect to $G$, and a determinant is indicated by writing a typical row within parentheses. The forms $F_2$, $\Phi_3$ are of the first and second orders respectively, and are both absolutely invariant under transformation of curvilinear coordinates $u$, $v$ on $S$. But under transformation of fundamental form $G$ and of proportionality factor $\rho$ of the homogeneous coordinates, the forms $F_2$, $\Phi_3$ behave quite differently.

Let us define a differential form $F_3$ by the equation

$$F_3 = 2\Phi_3 - 3dF_2 + \frac{3}{4}F_2 d\log\frac{\Delta}{A}, \tag{11}$$

where $\Delta$ is the discriminant of $F_2$. Both forms $F_2$, $F_3$ are of the first order of degrees indicated by the subscripts, are absolutely invariant under transformation of curvilinear coordinates, and are cogredient under transformation of $G$ and $\rho$. In fact both forms are relative invariants under the latter transformation, since we have

$$|A'|^{1/2}F_i' = \rho^4|A|^{1/2}F_i, \qquad (i = 2,3), \tag{12}$$

accents indicating the transformed quantities.

Let us define two forms $\phi_2$, $\phi_3$ by the formula

$$\phi_i = \rho^4|A|^{1/2}|A'|^{-1/2}F_i, \qquad (i = 2,3), \tag{13}$$

wherein $\rho$ is chosen so that the ratio of the discriminant of $\phi_3$ to the cube of the discriminant of $\phi_2$ is constant. With this choice of $\rho$ the coordinates are *Fubini's normal coordinates*. Next let us choose the fundamental form $G$ to be $\phi_2$. Then the hessian of $\phi_3$ is proportional to $\phi_2$, so that $\phi_3$ is apolar to $\phi_2$. If the asymptotic net on $S$ is taken for parametric net, we can write

$$\phi_2 = 2\beta\gamma du dv, \quad \phi_3 = 2\beta\gamma(\beta du^3 + \gamma dv^3). \tag{14}$$

If we attempt to set up the differential equations to obtain a surface which has the forms $\phi_2$, $\phi_3$ arbitrarily assigned, we find that $\phi_2$, $\phi_3$ are not sufficient to define a surface to within a projective transformation. But if we adjoin to $\phi_2$, $\phi_3$ the form $\psi_2$ defined by

$$\psi_2 = pdu^2 + qdv^2, \tag{15}$$

then the forms $\phi_2$, $\phi_3$, $\psi_2$ do define a surface, except for a projective transformation, and the differential equations for obtaining such a surface are precisely equations (1). At this point the Italian and American theories merge.

The form $\phi_2$ vanishes for the asymptotic curves, and $\phi_3$

for the curves of Darboux, while $\psi_2$ vanishes for the curves corresponding to the developables of the congruence of lines which are polars of the projective normals with respect to the quadrics of Lie.

6. *Extensions to Hyperspace.* Extensive investigations in the projective differential geometry of hyperspace have been made by Segre[32], Bompiani, Fubini, Terracini, Tzitzéica[33] and others. We shall confine our attention to only two of the many topics that might be considered, first discussing Bompiani's *pluri-axial systems*, and then comparing the Italian and American methods as applied in hyperspace.

7. *Pluri-Axial Systems.* In a linear space $S_n$ of $n$ dimensions let us consider a surface $S$ which has on it a conjugate net, so that at every point of $S$ the osculating planes of all the curves on $S$ through the point are contained in an $S_4$ instead of in the usual $S_5$. Let a congruence $\Gamma$ of planes be given, one of which passes through each point $P$ of $S$ and lies in the tangent $S_4$ without intersecting, except at $P$, the tangent plane of $S$ at $P$. Then a curve on $S$ is *a planar curve*[34] *of* $\Gamma$ in case its osculating plane at each of its points intersects in a straight line the plane of $\Gamma$ through the point. The differential equation of a planar system of curves has the form (6). It is well known that point geometry on a hyperquadric in $S_5$ is isomorphic with line geometry in ordinary ruled space $R_3$, the correspondance being established by interpreting the six Plückerian coordinates of a line in $R_3$ as the coordinates of a point in $S_5$. If the surface $S$ under consideration here is the surface in $S_5$ that corresponds to one congruence of asymptotic tangents of a surface in $R_3$, then the planar system of curves defined here corresponds to the planar system previously defined.

On a general surface $S$ in $S_n$ a system of curves analogous to a planar system can be defined[35] by considering a congruence of spaces $S_3$, instead of planes, one of which passes through each point $P$ of $S$ and lies in the tangent $S_5$ at $P$. More generally, let us consider at each point $P$ of $S$

the $S(k)$ which is the ambient space of the osculating $S_k$ at $P$ of every curve on $S$ through $P$. Then let a congruence $\Gamma$ of linear spaces $S_h$ be given, one of which passes through each point $P$ of $S$ and lies in the $S(k)$ at $P$. Then a curve on $S$ is *a pluri-axial curve of* $\Gamma$ in case its osculating $S_k$ at each of its points intersects the $S_h$ of $\Gamma$ through the point in a linear space of dimensions greater than would ordinarily be the case.

Still further generalizations can be obtained[36] by replacing the surface $S$ by a variety $V_k$ of $k$ dimensions and associating with each point of $V_k$ a linear space $S_h$. Or instead of linear spaces $S_h$, cones may be used. And instead of a pluri-axial system of curves, it would seem possible to define a pluri-axial system of varieties $V_j$ on $V_k$ $(1<j<k)$.

8. *Comparison of Italian and American Methods.* The analytic and synthetic methods used by Italian geometers in studying the projective differential geometry of hyperspace possess great power and elegance. Nevertheless there does not yet exist a general theory of a $V_k$ in $S_n$ comparable with the Italian and American theories of a surface in ordinary space. The Italian method of differential forms has failed for a $V_k$ in $S_n$ $(1<k<n-1,\ n>4)$, either because of the lack of a covariant quadratic differential form, or because of the lack of an absolute calculus for an $n$-ary $p$-adic differential form. The American method of differential equations is theoretically available, but the amount of labor involved in making the necessary calculations seems at present to be practically prohibitive in so far as establishing a general theory is concerned. In special cases, however, the American method is applicable. For instance, Miss Beenken has recently[37] applied this method in a study of surfaces in $S_5$. And there is hope that it may be possible to modify the American method so as to avoid much of the labor of the calculations ordinarily involved by reducing all of the differential equations of a defining system to equations of the first order[38].

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# FACTORIAL SERIES IN TWO VARIABLES*

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Because of its connection with the study of partial difference equations, the author has had occasion to examine the factorial series

$$\text{(A)} \qquad \sum_{n=0}^{\infty} \frac{(n!)^2 a_n}{x(x+1)\cdots(x+n)y(y+1)\cdots(y+n)}.$$

It may be of some interest to observe that the methods employed by Landau in his fundamental paper on factorial series† can immediately be extended to deduce the essential convergence properties of the series (A). We give here eight theorems corresponding to Theorems I–VIII of Landau. They may all be established by proofs parallel to his in every detail; some of them may also be inferred otherwise as indicated below.

THEOREM 1. *If the series* (A) *converges for the place*‡ $(x_0, y_0)$, *it converges for any place* $(x_1, y_1)$ *satisfying the conditions*§ $R(x_1) \geqq R(x_0)$, $R(y_1) \geqq R(y_0)$ (*both equality signs not to hold simultaneously*).

Only the following three possibilities may therefore occur: (a) the series (A) converges everywhere; (b) it converges nowhere; (c) there exist two associated real numbers $\lambda_1$, $\lambda_2$, such that the series converges for $R(x) \geqq \lambda_1$, $R(y) \geqq \lambda_2$ and diverges for $R(x) \leqq \lambda_1$, $R(y) \leqq \lambda_2$ (both equality signs not to hold simultaneously in either case). One would scarcely

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*Presented to the Society, April 6, 1928.

† *Über die Grundlagen der Theorie der Fakultätenreihen*, Sitzungsberichte der Münchener Akademie (math.-phys.), vol. 36 (1906), pp. 151-218.

‡We think of the complex variables $x$ and $y$ as represented by points in two distinct planes. Following a common terminology a pair of values $x$, $y$ will be spoken of as the *place* $(x, y)$.

§$R(x)$ is used to denote the *real part* of $x$.

expect the pair of numbers $\lambda_1, \lambda_2$ to be uniquely determined; the relation between them will presently be evident.

THEOREM 2. *The series* (A) *converges uniformly in the neighborhood of every place* $(x, y)$ *satisfying the conditions* $R(x) > \lambda_1$; $R(y) > \lambda_2$; *and* $x, y \neq 0, -1, -2, \cdots$.

Then from a well known theorem in the theory of functions* follows

THEOREM 3. *The series* (A) *represents an analytic function within its related half-planes of convergence* $(x, y \neq 0, -1, -2, \cdots)$ *and in this region may be differentiated partially with respect to either variable as many times as may be desired.*

THEOREM 4. *The region of absolute convergence of the series* (A), *unless it converges everywhere or nowhere, is a pair of related half-planes bounded on the left by lines* $R(x) = \mu_1$, $R(y) = \mu_2$; *the lines themselves may be included or not.*

THEOREM 5. *If the series* (A) *converges for the place* $(x_0, y_0)$ *and if we have* $R(x_1) > R(x_0) + a$, $R(y_1) > R(y_0) + b$, *and* $a + b = 1$, *the series* (A) *is absolutely convergent for the place* $(x_1, y_1)$.

THEOREM 6. *The regions of convergence of the series* (A) *and of the Dirichlet series*

$$\text{(B)} \qquad \sum_{n=1}^{\infty} \frac{a_n}{n^{x+y}}$$

*are the same; that is, for any place* $(x, y)$ $(x, y \neq 0, -1, -2, \cdots)$ *both series converge or both diverge.*

From this theorem, which is proved independently of the preceding ones, Theorem 1 could be inferred. In conjunction with Landau's Theorem VI it also forms the basis for the rather surprising result that *the regions of convergence of* (A) *and of the series*

$$\sum_{n=0}^{\infty} \frac{n! a_n}{(x+y)(x+y+1)\cdots(x+y+n)}$$

*See Osgood, *Lehrbuch der Funktionentheorie*, vol. II, 1924, p. 13.

*are identical.* Hence we conclude that *the related abscissas of convergence of the series* (A) *have a constant sum.*

THEOREM 7. *The places of absolute convergence of the series* (A) *and* (B) *are the same.*

From Theorems 6 and 7 and Landau's Theorems VI–VIII follows at once

THEOREM 8. *If the sum of the abscissas of* [*absolute*] *convergence* $\lambda_1+\lambda_2[\mu_1+\mu_2]$ *is* $\geqq 0$, *it is given by*

$$\limsup_{t\to\infty} \frac{\log\left|\sum_{n=1}^{t} a_n\right|}{\log t} \left[\limsup_{t\to\infty} \frac{\log \sum_{n=1}^{t} |a_n|}{\log t}\right].$$

It may be remarked that Landau's work can similarly be extended to prove the convergence properties of the series

$$\sum_{n=0}^{\infty} (n!)^m a_n / [x_1(x_1+1)\cdots(x_1+n)x_2(x_2+1)\cdots(x_2+n)\cdots x_m(x_m+1)\cdots(x_m+n)],$$

where $m$ is any positive integer.

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# THE NODES OF THE RATIONAL PLANE QUARTIC*

BY L. T. MOORE

The projective properties of the rational quartic curve may be studied from two points of view. If the curve is given parametrically, its invariants are expressible in terms of the coefficients of the fundamental involution. If the curve is given by the plane $(KX)=0$, and the Steiner quartic surface $(X^{1/2})=0$, the invariants are expressible in symmetric functions of the coefficients of $(KX)=0$.† An invariant condition obtained for one form of the equations of the curve may readily be expressed in the invariants of the other form. The relations connecting the two systems are

$$(1)\qquad \begin{cases} 4I_2 = MS_1S_3, \\ I_2' = M(S_1S_3 - 16S_4), \\ I_4 = (16M)^2(S_2S_3^2 - S_1S_3S_4 + S_4^2), \\ I_6 = M^3S_3^2(S_1S_2S_3 - S_1^2S_4 - S_3^2), \end{cases}$$

where $M$ is a positive constant. These equations may also be solved for any function of $S_1$, $S_2$, $S_3$, $S_4$, whose weight is a multiple of four.‡

To determine the nature of the nodes from the invariants, it is desirable to refer the quartic to a triangle whose vertices are at the nodes. Such a quartic is given by the equations§

$$X_0^2X_1^2 + X_1^2X_2^2 + X_0^2X_2^2 - 2X_0X_1X_2X_3 = 0,$$

$$\alpha_0X_0 + \alpha_1X_1 + \alpha_2X_2 + \alpha_3X_3 = 0.$$

Eliminating $X_3$ we have

$$(2)\quad \alpha_2(X_0^2X_1^2 + X_1^2X_2^2 + X_0^2X_2^2) + 2X_0X_1X_2(\alpha_0X_0 + \alpha_1X_1 + \alpha_2X_2) = 0,$$

* Presented to the Society, May 7, 1927.

† J. E. Rowe, Transactions of this Society, vol. 12 (1911), pp. 295–310.

‡ L. T. Moore, American Journal, vol. 48 (1926), p. 251.

§ The surface $(\sqrt{X})=0$ referred to a tetrahedron having the three double lines as edges. Salmon-Rogers, *Geometry of Three Dimensions*, vol. 2, p. 213.

which may be derived from the conic

$$(3)\quad \alpha_3(X_0^2 + X_1^2 + X_2^2) + 2\alpha_2X_0X_1 + 2\alpha_1X_0X_2 + 2\alpha_0X_1X_2 = 0,$$

by the transformation $X_0 = 1/X_0'$; $X_1 = 1/X_1'$; $X_2 = 1/X_2'$.

The points of intersection of (3) with the sides of the triangle of reference correspond to the nodes of (2). Therefore, if (3) cuts a side in two real, coincident, or imaginary points, the corresponding node is, respectively, a crunode, a cusp, or an acnode.*

It is evident that the expressions $(\alpha_0^2 - \alpha_3^2)$, $(\alpha_1^2 - \alpha_3^2)$, $(\alpha_2^2 - \alpha_3^2)$ determine the nature of the nodes, and from a consideration of the functions

$$\begin{aligned} F_3(\alpha) &= (\alpha_0^2 - \alpha_3^2)(\alpha_1^2 - \alpha_3^2)(\alpha_2^2 - \alpha_3^2), \\ F_2(\alpha) &= (\alpha_0^2 - \alpha_3^2)(\alpha_1^2 - \alpha_3^2) + (\alpha_0^2 - \alpha_3^2)(\alpha_2^2 - \alpha_3^2) \\ &\quad + (\alpha_1^2 - \alpha_3^2)(\alpha_2^2 - \alpha_3^2), \\ F_1(\alpha) &= (\alpha_0^2 - \alpha_3^2) + (\alpha_1^2 - \alpha_3^2) + (\alpha_2^2 - \alpha_3^2), \end{aligned}$$

the following classification can be obtained:

*Three crunodes*: $F_3(\alpha) > 0$, $F_2(\alpha) > 0$, $F_1(\alpha) > 0$;

*Two crunodes, one acnode*: $F_3(\alpha) < 0$, $\{F_1(\alpha) > 0$ or $F_1(\alpha) < 0$, $F_2(\alpha) < 0\}$;

*Three acnodes*: $F_3(\alpha) < 0$, $F_2(\alpha) > 0$, $F_1(\alpha) < 0$;

*One crunode, two acnodes*: $F_3(\alpha) > 0$, $F_2(\alpha)$ or $F_1(\alpha) < 0$.

The following relations connect the coefficients of $(\alpha X) = 0$ and $(KX) = 0$:†

$$\begin{aligned} S_1 &= K_0 + K_1 + K_2 + K_3 = 4\alpha_3, \\ S_2 &= K_0K_1 + K_0K_2 + K_0K_3 + K_1K_2 + K_1K_3 + K_2K_3 \\ &= 2(3\alpha_3^2 - \alpha_0^2 - \alpha_1^2 - \alpha_2^2), \\ S_3 &= 4(\alpha_3^3 - \alpha_3(\alpha_0^2 + \alpha_1^2 + \alpha_2^2)) + 2\alpha_0\alpha_1\alpha_2), \\ S_4 &= \alpha_0^4 + \alpha_1^4 + \alpha_2^4 + \alpha_3^4 - 2\alpha_1^2\alpha_0^2 - 2\alpha_0^2\alpha_2^2 - 2\alpha_0^2\alpha_3^2 \\ &\quad - 2\alpha_1^2\alpha_2^2 - 2\alpha_1^2\alpha_3^2 - 2\alpha_2^2\alpha_3^2 + 8\alpha_0\alpha_1\alpha_2\alpha_3. \end{aligned}$$

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* Salmon, *Higher Plane Curves*, 1879, p. 254.

† L. T. Moore, loc. cit., p. 248.

From these relations we have

$$64F_3(\alpha) = (S_1^2S_4 - S_1S_2S_3 + S_3^2),\ 16F_2(\alpha) = (S_2^2 + S_1S_3 - 4S_4),$$
$$2F_1(\alpha) = -S_2.$$

Again, after multiplying $F_1(\alpha)$ by $S_1^2$ and $F_3(\alpha)$ by $S_3^2$, we have from equations (1),

$$M^2S_3^2(S_1^2S_4 - S_1S_2S_3 + S_3^2) = -I_6,$$

$$1024M(S_2^2 + S_1S_3 - 4S_4)$$
$$= \frac{\{I_4 + 64I_2(4I_2 - I_2') - (4I_2 - I_2')^2\}^2 + 256I_6'(I_2' + 12I_2)}{I_6'},$$

$$MS_1^2S_2 = \frac{4I_2^3\{I_4 + 64I_2(4I_2 - I_2') - (4I_2 - I_2')^2}{I_6'},$$

where $I_6' = I_2I_4 - I_2(I_2' - 4I_2)^2 - 64I_6$.

Thus we have established sufficient criteria, in each of the three systems of invariants, to determine the nature of the nodes of the curve.

We can determine the nature of the nodes of rational tacnodal and oscnodal quartics by the same method used for trinodal quartics.* The results obtained may be stated as follows:

*Tacnode and crunode*: $I_6 < 0$, $I_2' - 4I_2 < 0$;
*Tacnode and acnode*: $I_6 > 0$, $I_2' - 4I_2 < 0$;
*Isolated tacnode and crunode*: $I_6 < 0$, $I_2' - 4I_2 > 0$;
*Isolated tacnode and acnode*: $I_6 > 0$, $I_2' - 4I_2 > 0$;
*Isolated oscnode*: $I_6 < 0$.

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* L. T. Moore and J. H. Neelley, *Rational tacnodal and oscnodal quartic curves considered as plane sections of quartic surfaces*, to appear soon in the American Journal. In this paper, equations corresponding to equations (1), (2), (3) above are developed for tacnodal and oscnodal quartics.

# A THEOREM ON RULED SURFACES

BY A. F. CARPENTER

On page 229 of volume 1 of Fubini-Čech, *Geometria Proiettiva Differenziale*, appears the following theorem.

*If the two flecnode curves of a ruled surface are plane, then the complex curves and the harmonic curves are plane, and all of their planes belong to a pencil.*

The authors furnish a proof of this, and they call it "an interesting theorem by Sullivan."

In a letter to the author of this note, Professor C. T. Sullivan says, with reference to the above, "The theorem you refer to is consequently not to be found in any of my publications." It appears probable that the Italian geometers have confused this theorem with one of a somewhat similar nature published by the author in 1915,* or perhaps with a somewhat more general theorem contained in a later paper by the author.†

In this paper it is shown that the planes osculating the two branches of the flecnode curve and the planes osculating the two branches of the complex curve at the four points in which these curves cut a line element of their supporting ruled surface, will form a pencil, if and only if,

$$D \equiv p_{21}^{2}\Delta_1 - p_{12}^{2}\Delta_2 = 0, \tag{1}$$

where

$$\Delta_1 = p_{12}^{2}q_{22} - p_{12}q_{12}' + 3q_{12}^{2}, \quad \Delta_2 \equiv p_{21}^{2}q_{11} - p_{21}q_{21}' + 3q_{21}^{2},$$

$p_{ik}$, $q_{ik}$ being coefficients of the flecnode canonical form of the system of differential equations defining the surface. The theorem of 1915 is a special case of that of 1923 since

*Transactions of this Society, vol. 16 (1915), p. 520. This volume also contains a paper by Professor Sullivan.

†Tôhoku Mathematical Journal, vol. 23 (1923), p. 114.

$\Delta_1=0$, $\Delta_2=0$ are necessary and sufficient conditions that the flecnode curves be plane.

It is the further purpose of this note to extend the theorem of the Tôhoku paper to include the harmonic curves.*

In the notation there employed the harmonic points are given by the factors of the covariant $p_{21}y^2+p_{12}z^2$. If $\eta=(p_{21})^{1/2}y+i(p_{12})^{1/2}z$, $\zeta=(p_{21})^{1/2}y-i(p_{12})^{1/2}z$, then the equations of the two planes osculating the curves $C_\eta$, $C_\zeta$, at points $P_\eta$, $P_\zeta$, are respectively

$$(2)\quad p_{21}(p_{12})^{1/2}(p_{12}p_{21}(p_{12}p_{21})^{1/2}+iP)(i(p_{12})^{1/2}x_1-(p_{21})^{1/2}x_2)+f_1(i)x_3+f_2(i)x_4=0,$$

$$(3)\quad p_{21}(p_{12})^{1/2}(p_{12}p_{21}(p_{12}p_{21})^{1/2}-iP)(i(p_{12})^{1/2}x_1+(p_{21})^{1/2}x_2)-f_1(-i)x_3-f_2(-i)x_4=0,$$

where

$$f_1(i)\equiv D+2p_{12}q_{21}P-p_{12}^3p_{21}^3-ip_{12}p_{21}(p_{12}p_{21})^{1/2}(2p_{12}q_{21}+P),$$

$$(p_{12})^{1/2}f_2(i)\equiv i(p_{21})^{1/2}(D+2p_{21}q_{12}P-p_{12}^3p_{21}^3)+p_{12}p_{21}^2(p_{12})^{1/2}(2p_{21}q_{12}-P),$$

$$P=p_{12}q_{21}-p_{21}q_{12}.$$

Two of the pencil of planes determined by (2), (3) are seen to be

$$(4)\quad (p_{12}^3p_{21}^3+P^2)(p_{12}x_2+p_{12}^2x_3-2q_{12}x_4)-p_{12}D(p_{12}^2p_{21}x_3-Px_4)=0,$$

$$(5)\quad (p_{12}^3p_{21}^3+P^2)(p_{21}x_1-2q_{21}x_3+p_{21}^2x_4)-p_{21}D(Px_3-p_{12}p_{21}^2x_4)=0.$$

If, and only if, $D=0$, will (4), (5) reduce to

$$(6)\quad p_{12}x_2+p_{12}^2x_3-2q_{12}x_4=0,$$

$$(7)\quad p_{21}x_1-2q_{21}x_3+p_{21}^2x_4=0.$$

But (6) and (7) are the equations of the planes osculating

*These curves were called by Wilczynski the *involute curves*. See his *Projective Differential Geometry*, p. 208.

the flecnode curves. Combining this result with the theorem of the Tôhoku paper we have the following theorem.

THEOREM. *The planes osculating the flecnode curve, the complex curve and the harmonic curve at the six points in which these curves cut a line element of their supporting ruled surface, will belong to a pencil, if, and only if,*

$$D \equiv p_{21}^{2}\Delta_1 - p_{12}^{2}\Delta_2 = 0.$$

Since, when $\Delta_1 = \Delta_2 = 0$, the flecnode curves are plane, the theorem of Fubini-Čech appears as a special case of the preceding theorem.

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## NETS OF CONICS IN THE GALOIS FIELDS OF ORDER $2^n$*

BY A. D. CAMPBELL

We consider a net of conics in the Galois fields of order $2^n$

$$\lambda C_1 + \mu C_2 + \nu C_3 = 0, \tag{1}$$

where

$$C_i \equiv a_i x^2 + b_i y^2 + c_i z^2 + f_i yz + g_i zx + h_i xy, \quad (i = 1,2,3),$$

and where $C_i$ has the discriminant†

$$\Delta_i \equiv f_i g_i h_i + a_i f_i^2 + b_i g_i^2 + c_i h_i^2 .$$

Such a field is denoted for brevity by the symbol $GF(2^n)$. Along with (1) we consider the cubic curve in $\lambda$, $\mu$, $\nu$ that is obtained by equating to zero the discriminant of the general conic of (1)‡

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* Presented to the Society, December 31, 1926.

† See A. D. Campbell, *Plane cubic curves in the Galois fields of order* $2^n$, Annals of Mathematics, vol. 27 (1926), p. 395.

‡Compare C. Jordan, *Réduction d'un réseau de formes quadratiques*, Journal des Mathématiques, (6), vol. 2 (1906), p. 412.

$$(2)\quad K \equiv a\lambda^3 + b\mu^3 + c\nu^3 + d\lambda^2\mu + e\lambda\mu^2 + f\lambda^2\nu + g\lambda\nu^2 + h\mu^2\nu + j\mu\nu^2 + k\lambda\mu\nu = 0,$$

where $a \equiv \Delta_1$, $b \equiv \Delta_2$, $c \equiv \Delta_3$, and

$$d \equiv f_1g_1h_2 + f_1g_2h_1 + f_2g_1h_1 + a_2f_1^2 + b_2g_1^2 + c_2h_1^2,$$

with similar expressions for $e, f, g, h, j$; and where

$$k \equiv f_1g_2h_3 + f_1g_3h_2 + f_2g_1h_3 + f_2g_3h_1 + f_3g_1h_2 + f_3g_2h_1.$$

In this paper we shall classify these nets of conics, deriving a typical net for each class. First we shall study the nets that have no degenerate pencil of conics* and hence have a non-composite cubic $K=0$; then we shall consider the nets with composite cubics. For the first type of nets we reduce the cubic to a standard form and then put the net in a canonical form.

If the net (1) has a double line this must correspond to a cusp on the cubic (2), a fact easy to prove. We put (2) in the form†

$$\lambda^2\mu + \lambda^2\nu + \mu\nu^2 = 0$$

with a cusp at (0, 1, 0) and an inflection at (1, 0, 0). To the tangent at the inflection on the cubic there must correspond in the net a pencil with just one degenerate conic $C$ which corresponds to the inflection. This conic $C$ must be a real line pair, because there is no such pencil with $C$ a pair of conjugate imaginary lines.‡ We can now put (1) in the form

$$\lambda yz + \mu x^2 + \nu C_3 = 0,$$

$$K \equiv \lambda^2\mu + a_3\lambda^2\nu + g_3h_3\lambda\nu^2 + f_3^2\mu\nu^2 + \Delta_3\nu^3 = 0.$$

We must have $a_3=1$, $g_3h_3=0$, $f_3^2=1$, $\Delta_3=0$; hence $h_3=0$ or $g_3=0$. We can assume that $h_3=0$, because the transformation

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* That is, a pencil with $\Delta\equiv 0$.

†Campbell, loc. cit., p. 398. See this paper also for the other typical forms of cubics.

‡See A. D. Campbell, *Pencils of conics in the Galois fields of order* $2^n$, American Journal of Mathematics, vol. 49 (1927), pp. 401-406 for such facts and for typical pencils of conics.

$x=x'$, $y=z'$, $z=y'$ reduces the case $g_3=0$ to the case $h_3=0$. Therefore we have $h_3=0$, $c_3g_3^2+1=0$. If $c_3\neq 0$, we put*

$$\lambda = \lambda' + \nu', \quad \mu = \mu' + \nu', \quad \nu = \nu';$$

$$x = \frac{1}{g_3}x', \qquad y = g_3y', \qquad z = \frac{1}{c_3^{1/2}}z';$$

then

$$\lambda' = \frac{c_3^{1/2}}{g_3}\lambda'', \quad \mu' = g_3^2\mu'', \quad \nu' = \nu'';$$

and after dropping the primes and double primes from variables and parameters we obtain

$$(3) \qquad \lambda yz + \mu x^2 + \nu(y^2 + z^2 + zx) = 0, \quad K \equiv \lambda^2\mu + \nu^3 = 0,$$

whose cubic is reducible to the above typical cubic. If $c_3=0$, we get

$$(4) \qquad \lambda yz + \mu x^2 + \nu(y^2 + zx) = 0, \quad K \equiv \lambda^2\mu + \nu^3 = 0.$$

The cubics of these two nets are the same, but any transformation that is to send (3) into (4) must send the pencil $\nu=0$ of (3) into the same pencil of (4)† and so must have the form

$$T : \rho x = \alpha_1 x', \quad \rho y = \beta_2 y', \quad \rho z = \gamma_3 z';$$

or

$$\rho x = \alpha_1 x', \quad \rho y = \gamma_2 z', \quad \rho z = \beta_3 y'.$$

However, $T$ sends (3) into a net containing the term in $z^2$, hence (3) and (4) are non-equivalent.

Next we suppose the cubic (2) to have a crunode and no real inflection, and we reduce the cubic to the form

$$\alpha\lambda^3 + \mu^3 + \lambda\mu\nu = 0,$$

with crunode at (0, 0, 1), where $\alpha\neq$ cube and $2^n=3m+1$. As we discussed above the sort of conic in the net that must

* We call this change of parameters "using $C_1$ to rid $C_3$ of $yz$, and using $C_2$ to rid $C_3$ of $x^2$."

†Because the double line has to go into a double line, also $\nu=0$ in the $(\lambda, \mu, \nu)$ plane joins the cusp to the inflection on the cubic and so must go into the same line on the other cubic.

correspond to an inflection on the cubic so here we can readily show that to the crunode there must correspond a real line pair in the net. We put our net in the form

$$\lambda C_1 + \mu(y^2 + zx) + \nu xy = 0,\ K^* \equiv \Delta_1\lambda^3 + (fh + g^2)\lambda^2\mu + b\lambda\mu^2 + fg\lambda^2\nu + c\lambda\nu^2 + f\lambda\mu\nu + \mu^3 = 0.$$

We must have $b=c=g=0$, $f=1$, $h=0$, $a=\alpha\neq$cube; hence we get

$$\lambda(\alpha x^2 + yz) + \mu(y^2 + zx) + \nu xy = 0. \tag{5}$$

Now we consider the cubic (2) to have a crunode and just one real inflection, and put the cubic in the form

$$\lambda^3 + \mu^2\nu + \lambda\mu\nu = 0,$$

with crunode at (0, 0, 1) and inflection at (0, 1, 0), which cubic exists only in the $GF(2)$. We reduce the net to the form

$$\lambda(y^2 + zx) + \mu C_2 + \nu xy = 0,\quad K \equiv \lambda^3 + b\lambda^2\mu + (fh + g^2)\lambda\mu^2 + f\lambda\mu\nu + fg\mu^2\nu + c\mu\nu^2 + \Delta_2\mu^3 = 0.$$

We have $b=0$, $f=g=h=a=1$, $c=0$. We use $C_3$ to rid $C_2$ of $xy$ and we get

$$\lambda(y^2 + zx) + \mu(x^2 + yz + zx) + \nu xy = 0. \tag{6}$$

If for a net (1) the cubic (2) has a crunode and at least two real inflections, we put this cubic in the form

$$\lambda^2\mu + \lambda\mu^2 + \lambda^2\nu + \mu^2\nu + \lambda\mu\nu = 0,$$

existent only in $GF(2^n)$ for $n>1$, with crunode at (0, 0, 1) and inflections at (1, 0, 0) and (0, 1, 0). The net can be written

$$\lambda C_1 + \mu xy + \nu(z^2 + zx) = 0,\quad K \equiv \Delta_1\lambda^3 + fg\lambda^2\mu + c\lambda\mu^2 + (fh + h^2)\lambda^2\nu + b\lambda\nu^2 + \mu^2\nu + f\lambda\mu\nu = 0.$$

We find that $fg=c=fh+h^2=1$, $b=0$, $f=1$, $\Delta_1\equiv h+a+h^2=0$;

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* We shall from now on omit the subscripts from such coefficients as $f_1$, $h_1$, etc., and $a_2$, $b_2$, etc., and $a_3$, $b_3$, etc.

hence $g=h^2+h=a=1$. We use $C_2$ to rid $C_1$ of $xy$ and $C_3$ to rid $C_1$ of $z^2+zx$, and we obtain

$$\lambda(x^2+yz)+\mu xy+\nu(z^2+zx)=0. \tag{7}$$

We consider the net (1) to have a cubic (2) with an acnode and no real inflections, and we reduce the cubic to the form

$$\lambda^2\mu+\lambda^2\nu+\alpha\mu^2\nu+\lambda\mu\nu=0,$$

with acnode at (0, 0, 1) and $\lambda^2+\alpha\mu^2+\lambda\mu=0$ as tangents. If we suppose that to an acnode there corresponds a real line pair in the net we are led to a contradiction, hence to the acnode there must correspond a conjugate imaginary line pair. Therefore we can put (1) in the form

$$\lambda C_1+\mu xy+\nu(x^2+z^2+\alpha zx)=0, \tag{$n$}$$
$$K\equiv\Delta_1\lambda^3+fg\lambda^2\mu+c\lambda\mu^2+(h^2+f^2+\alpha fh)\lambda^2\nu$$
$$+\alpha^2b\lambda\nu^2+\mu^2\nu+\alpha f\lambda\mu\nu=0.$$

We must have $c=b=\Delta_1=0$, $fg=\alpha f=h^2+f^2+\alpha fh$; hence $g=\alpha$, $fgh+af^2\equiv\alpha fh+af^2=0$. We use $C_2$ to rid $C_1$ of $xy$, and put

$$\lambda=\frac{f}{\alpha h}\lambda',\quad \mu=\frac{f^2}{\alpha h}\mu',\quad \nu=\nu';\ x=x',\ y=\frac{\alpha h}{f^2}y',\ z=z'$$

and we get

$$\lambda(x^2+yz+fzx/h)+\mu xy+\nu(x^2+z^2+\alpha zx)=0, \tag{8}$$

where $h^2+f^2+\alpha fh+\alpha f=0$.

We deal with a net whose cubic has an acnode and just one real inflection, and we put this cubic in the form

$$\lambda^3+\lambda^2\nu+\alpha\mu^2\nu+\lambda\mu\nu=0,$$

which is nonexistent in $GF(2)$, with acnode at (0, 0, 1) and inflection at (0, 1, 0). We can put the net in the above form ($n$) with its cubic $K=0$. Hence we see that $fg=b=c=0$, $\alpha f=h^2+f^2+\alpha fh=\Delta_1\neq 0$; therefore $g=0, f\neq 0, af^2=\alpha f, a=\alpha/f$. We use $C_2$ to rid $C_1$ of $xy$, then put

$$\lambda = \frac{f}{\alpha}\lambda', \quad \mu = \frac{f^2}{\alpha}\mu_1', \quad \nu = \nu' ; \quad x = x', \quad y = \frac{\alpha}{f^2}y', \quad z = z'$$

and we obtain

$$\lambda(x^2 + yz) + \mu xy + \nu(x^2 + z^2 + \alpha zx) = 0. \tag{9}$$

Next we study the net (1) when it has a cubic (2) with an acnode and at least two real inflections, and we reduce this cubic to the form

$$\lambda^2\mu + \lambda\mu^2 + \lambda^2\nu + \mu^2\nu + \lambda\mu\nu = 0,$$

existent only in $GF(2)$, with acnode at (0, 0, 1) and one real inflection at (0, 1, 0). We can put our net in the above form $(n)$ with $\alpha = 1$. We must have $fg = c = h^2 + f^2 + fh = f = 1$, $b = 0$, $\Delta_1 = 0$; hence $f = g = c = 1$, $h = 1$ and $a = 0$, or $h = 0$ and $a = 0$. We use $C_2$ to rid $C_1$ of $xy$ and we get

$$\lambda(z^2 + yz + zx) + \mu xy + \nu(x^2 + z^2 + zx) = 0. \tag{10}$$

Now we assume that the net (1) has a cubic (2) with no node or cusp and no real inflection, and we put this cubic in the form

$$\lambda^2\mu + \lambda\mu^2 + \alpha\lambda\nu^2 + \beta\mu\nu^2 + \lambda\mu\nu = 0.$$

The point (0, 1, 0) on this cubic was any point on the cubic, so we take a point that corresponds to a real line pair in the net, and put our net in the form

$$\lambda C_1 + \mu(z^2 + zx) + \nu xy = 0, \qquad K \equiv \Delta_1\lambda^3 + (fh + h^2)\lambda^2\mu + b\lambda\mu^2 + fg\lambda^2\nu + c\lambda\nu^2 + \mu\nu^2 + f\lambda\mu\nu = 0.$$

We see we must have $\Delta_1 = 0$, $f \neq 0$, $fg = 0$, $b = fh + h^2 = f \neq 0$; hence $g = 0$, $\Delta_1 \equiv af^2 + ch^2 = 0$. We can therefore reduce our net to the form

$$\lambda\left[\frac{c(h^2 + 1)(h + 1)}{h^4}x^2 + y^2 + \frac{c(h + 1)}{h^2}z^2 + yz\right] + \mu(z^2 + zx) + \nu xy = 0, \tag{11}$$

with a cubic that is transformable into the above typical cubic with $\alpha = c(h+1)/h^2$, $\beta = (h+1)/h^2$.

We consider the net (1) when its cubic (2) has no node or cusp and just one real inflection and can be put in the form

$$\lambda^2\mu + \lambda\mu^2 + \alpha\lambda\nu^2 + \mu^2\nu + \lambda\mu\nu = 0,$$

with inflection at (0, 1, 0). Since there are two real tangents to the cubic from the point (0, 0, 1), this point must correspond to a real line pair in the net.* We can reduce the net to the above form $(n)$. We have then $fg=c=1$, $fh+h^2=0$, $b=\alpha$, $f=1$, $\Delta_1=0$; hence $h=1$ or $h=0$, $g=1$, $a=\alpha$. We use $C_2$ to rid $C_1$ of $xy$ and get

$$\lambda(\alpha x^2 + \alpha y^2 + z^2 + yz + zx) + \mu xy + \nu(z^2 + zx) = 0. \tag{12}$$

We suppose the cubic (2) has no node or cusp but just one real inflection and can be put in the form

$$\beta\lambda^3 + \lambda^2\nu + \lambda\nu^2 + \alpha\mu^2\nu + \lambda\mu\nu = 0.$$

We reduce the net (1) to the form

$$\lambda(y^2 + zx) + \mu xy + \nu C_3 = 0, \quad K \equiv \lambda^3 + b\lambda^2\nu + (fh + g^2)\lambda\nu^2 + c\mu^2\nu + fg\mu\nu^2 + f\lambda\mu\nu + \Delta_3\nu^3 = 0.$$

This gives us $b=fh+g^2=f\neq 0$, $fg=0$, $\Delta_3=0$; hence $g=0$, $h=1$, $a=c/f^2$. By an obvious transformation on $\lambda$, $\mu$, $\nu$ alone, we get

$$\lambda(y^2 + zx) + \mu xy + \nu\left(\frac{c}{f^3}x^2 + y^2 + \frac{c}{f}z^2 + yz\right) = 0, \tag{13}$$

with cubic reducible to the above typical form with $\alpha=c/f$, $\beta=1/f$.

Next we consider the net (1) when its cubic (2) has no node or cusp but at least two real inflections, and reduce this cubic to the form

$$\alpha\nu^3 + \lambda^2\mu + \lambda\mu^2 + \lambda\mu\nu = 0,$$

with inflections at (0, 1, 0) and (1, 0, 0). We put our net in the form

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* This fact can be observed by noting the pencils of conics in the $GF(2^n)$ that have discriminants of the form $\lambda^3\mu$.

$$\lambda C_1 + \mu xy + \nu(y^2 + zx) = 0, \quad K \equiv \Delta_1\lambda^3 + fg\lambda^2\mu + c\lambda\mu^2 + (fh + g^2)\lambda^2\nu + b\lambda\nu^2 + \nu^3 + f\lambda\mu\nu = 0.$$

We must have $fg = c = f \neq 0$, $fh + g^2 = b = \Delta_1 = 0$; hence $g = 1$, $c = f$, $h = 1/f$, $a = (f+1)/f^3$. Using $C_2$ to rid $C_1$ of $xy$ we get

$$(14)\quad \lambda\left(\frac{f+1}{f^3}x^2 + fz^2 + fyz + zx\right) + \mu xy + \nu(y^2 + zx) = 0,$$

with cubic reducible to the above type form with $\alpha = 1/f$.

Finally we suppose our net (1) has no degenerate conic and therefore its cubic (2) is imaginary. We put our cubic in either one of the two forms

$$\lambda^3 + \alpha'\mu^3 + \beta'\nu^3 + \gamma'\lambda\nu^2 + \mu\nu^2 + \lambda\mu\nu = 0$$

or

$$\lambda^3 + \alpha'\mu^3 + \beta'\nu^3 + \lambda\mu\nu = 0,$$

where $\alpha'$ and $\beta'$ are non-cubes and $2^n = 3m + 1$. Then the net (1) can be put in the form

$$\lambda(z^2 + xy) + \mu(\alpha x^2 + \beta y^2 + yz) + \nu C_3 = 0,$$
$$K \equiv \lambda^3 + \alpha\mu^3 + c\lambda^2\nu + (fg + h^2)\lambda\nu^2 + \alpha\mu^2\nu + (gh + \alpha f^2 + \beta g^2)\mu\nu^2 + g\lambda\mu\nu + \Delta_3\nu^3 = 0.$$

We must have $c = fg + h^2 = 0$, $g \neq 0$. We put

$$x = \frac{1}{g}x' + \frac{f}{g}y', \quad y = y', \quad z = \frac{h}{g}y' + z',$$

and we get

$$\lambda(z^2 + b_1y^2 + h_1xy) + \mu(a_2x^2 + b_2y^2 + f_2xz) + \nu(a_3x^2 + b_3y^2 + zx) = 0,$$
$$K \equiv h_1^2\lambda^3 + b_1\lambda\nu^2 + a_2f_2^2\mu^3 + a_3f_2^2\mu^2\nu + b_2\mu\nu^2 + b_3\nu^3 + f_2h_1\lambda\mu\nu = 0.$$

This cubic must have one of the above forms, hence we see that $h_1^2 = f_2h_1 \neq 0$, $a_3 = 0$, so $f_2 = h_1 \neq 0$, and we get

$$(15)\quad \lambda(z^2 + \gamma y^2 + xy) + \mu(\alpha x^2 + \delta y^2 + yz) + \nu(\beta y^2 + zx) = 0,$$

where $\alpha\beta \neq 0$, $\alpha$ and $\beta$ non-cubes, $\gamma$ may be zero, $\delta = 1$ or 0.

We shall not derive the nets with degenerate cubics but merely list them below, because their derivation is so simple by the method employed by C. Jordan, loc. cit.

(16) $\lambda x^2 + \mu y^2 + \nu(z^2 + yz) = 0.$

(17) $\lambda x^2 + \mu y^2 + \nu yz = 0.$

(18) $\lambda x^2 + \mu y^2 + \nu(z^2 + xy) = 0.$

(19) $\lambda x^2 + \mu y^2 + \nu xy = 0.$

(20) $\lambda x^2 + \mu y^2 + \nu z^2 = 0.$

(21) $\lambda x^2 + \mu xy + \nu(z^2 + yz) = 0.$

(22) $\lambda x^2 + \mu xy + \nu yz = 0.$

(23) $\lambda x^2 + \mu xy + \nu(y^2 + z^2 + \alpha yz) = 0,$
$y^2 + z^2 + \alpha yz = 0$ irreducible.

(24) $\lambda x^2 + \mu xy + \nu(z^2 + zx) = 0.$

(25) $\lambda x^2 + \mu xy + \nu(y^2 + zx) = 0.$

(26) $\lambda x^2 + \mu xy + \nu xz = 0.$

(27) $\lambda x^2 + \mu(y^2 + xy) + \nu(z^2 + yz) = 0.$

(28) $\lambda x^2 + \mu(y^2 + xy) + \nu yz = 0.$

(29) $\lambda x^2 + \mu(y^2 + xy) + \nu(\alpha y^2 + z^2 + zx) = 0,$
$\mu^2 + \alpha\nu^2 + \mu\nu = 0$ irreducible.

(30) $\lambda x^2 + \mu(y^2 + xy) + \nu(z^2 + zx) = 0.$

(31) $\lambda xy + \mu xz + \nu(x^2 + y^2 + z^2 + \alpha yz) = 0.$

(32) $\lambda xy + \mu xz + \nu(y^2 + z^2 + \alpha yz) = 0.$

(33) $\lambda xy + \mu xz + \nu(x^2 + yz) = 0.$

(34) $\lambda xy + \mu xz + \nu yz = 0.$

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# CERTAIN CLASS-NUMBER RELATIONS IMPLIED IN THE NACHLASS OF GAUSS*

BY E. T. BELL

1. *Introduction.* In a previous note† it was pointed out that the fundamental expansions of the elliptic theta constant products of the third degree, due to Hermite and Kronecker, follow by a mere change of sign in an algebraic identity from sections 292-3 of the *Disquisitiones Arithmeticae* of Gauss. The posthumous notes of Gauss, on what would now be called the transformation of the elliptic theta functions, contain implicitly a great many further class-number relations of types much more abstruse than the famous eight of Kronecker. To justify this assertion we need translate only two of the simplest of Gauss' identities into their equivalents in terms of class numbers. These will be sufficient; several of the remaining identities give class-number relations of equal simplicity and elegance. The present relations are allied to some that were stated without any clue to their origin by Liouville,‡ also to others deduced by Gierster§ from the theory of modular correspondences. It is surprising that these simple and extremely elegant relations should have been overlooked, lying as they do on the very surface of Gauss' work. All are written down by the usual device of comparing coefficients in different expansions of the same functions. It will not be necessary to preserve the algebraic details beyond the statement of the relevant identities (those given by Gauss for the transformation of the third order), as all the technique is standard. The final formulas (1)–(18) have been verified numerically.

*Presented to the Society, San Francisco Section, April 7, 1928.

†This Bulletin, vol. 30 (1924), pp. 236-8.

‡For references, see Dickson's *History*, vol. 3, Chap. 6; also H. J. S. Smith, *Report on the Theory of Numbers.*

§Mathematische Annalen, vol. 21 (1883), p. 49.

2. *The Functions $A, B, C, H, K$.* If $\mu$ is an odd integer, we shall write $(-1|\mu)\equiv(-1)^{(\mu-1)/2}$, so that $(-1|-\mu)=-(-1|\mu)$. The functions $A, \cdots, K$ are defined as follows, where $m$ is an odd integer $>0$, and in each case the $\sum$ refers to all $\mu$ such that, for $m$ constant, the indicated (indeterminate) equation is satisfied; $\nu$ denotes an arbitrary (odd or even) integer $\gtrless 0$, and $\mu, \mu_1$ are odd integers $\gtrless 0$.

$$\begin{aligned}
m &= 16\nu + 3\mu^2 \ : \quad A(m) = \textstyle\sum(-1\,|\,\mu)\mu.\\
m &= 4\mu_1^2 + 3\mu^2 \ : \quad B(m) = \textstyle\sum(-1\,|\,\mu)\mu.\\
4m &= \mu_1^2 + 3\mu^2 \ : \quad C(4m) = \textstyle\sum(-1\,|\,\mu)\mu.\\
m &= 12\nu^2 + \mu^2 \ : \quad H(m) = \textstyle\sum(-1\,|\,\mu)\mu.\\
4m &= 3\mu_1^2 + \mu^2 \ : \quad K(4m) = \textstyle\sum(-1\,|\,\mu)\mu.
\end{aligned}$$

For example, if $m=19$, the values of $(\nu, \mu)$ from which $A(m)$ is computed are $(\nu, \mu)=(\pm 1, \pm 1)$, and hence $A(19)=4$; for $m=27$ we have $(\nu, \mu)$ $(0, \pm 3)$, and hence $A(27)=-6$.

For these we find the following theorems.

(1) $m \equiv 3 \bmod 8$ :

$$6A(m) = 3C(4m) = 2H(3m) = K(12m).$$

(2) $m \equiv 11 \bmod 24$ :

$$A(m) = C(4m) = H(3m) = K(12m) = 0.$$

(3) $m \equiv 23 \bmod 24$ :

$$B(m) = K(12m) = H(3m) = 0.$$

(4) $m \equiv 5 \bmod 6$ : $C(4m) = 0$.

(5) $m \equiv 7 \bmod 8$ :

$$2H(3m) = K(12m) = -6B(m).$$

(6) $m \equiv 3, 19 \bmod 24$ : $6A(m) = -K(4m)$.

Certain of these can be seen by inspection, on noting what numbers are not represented in the given forms; all, however, follow at once by the means indicated in §4.

3. *Class-Number Relations.* For uniformity we shall express all the relations in terms of $E$ alone. The notation

is that of H. J. S. Smith's *Report on the Theory of Numbers*; it agrees with Dickson's *History* (vol. 3, chap. 6); $F(n)$, $F_1(n)$ are respectively the numbers of odd, even classes for the negative determinant $-n$; $E(n) \equiv F(n) - F_1(n)$, so that $12\,E(n)$ is the total number of representations of $n$ as a sum of three squares. By convention $F(0) = 0$, $E(0) = 1/12$, and a class equivalent to $a(x^2+y^2)$ ($a \equiv$ an integer) contributes $1/2$ to $F$ or to $F_1$, while a class equivalent to $a(2x^2+2xy+2y^2)$ contributes $1/3$ to $F_1$. For expressing $E$ in terms of $F$ and $F_1$ there are the following well known relations, $(n \geqq 0)$,

$$3E(8n+3) = 2F(8n+3),\ E(8n+7) = 0,\ E(4n) = E(n),$$

$$E(4n+1) = F(4n+1),\ E(4n+2) = F(4n+2),\ F(4n) = 2F(n).$$

A convenient table for numerical verifications was given in a former paper.*

We shall call two relations *independent* if neither can be inferred from the other either by the theorems (1)–(6) of §2, or by the relations between $E$, $F$, $F_1$ stated above. The sense in which class-number relations are $n$-tuply infinite, $n > 0$, was defined and amply illustrated elsewhere†; briefly, a class-number relation involving arbitrary uniform functions, or arbitrary odd or even uniform functions of precisely $n$ independent variables, was called $n$-fold infinite; if no arbitrary function occurs in the relation, (we say that) the relation is *constant*. All the class-number relations reported in Dickson's *History* (loc. cit.) are constant in this sense; they all are degenerate cases of singly infinite relations. The relations next stated are equivalent to all those constant relations deducible from the transformation of the third order of the elliptic theta functions, combined with the transformation of the first order, and hence the corresponding Gauss formulas can give nothing further. It is to be particularly noted that we have included in this statement the transformations of the first order, as otherwise

* Tôhoku Mathematical Journal, vol. 19 (1921), p. 116.

† Quarterly Journal of Mathematics, vol. 49 (1923), pp. 322–337.

certain of the formulas may not be evident. The relations are not all independent in the sense above defined; we have given the dependent ones on account of their relative simplicity. The set is as follows. The $\sum$ refers to all integers $\nu \gtrless 0$, or to all odd integers $\mu \gtrless 0$, such that the arguments of the summands are $\geqq 0$.

(7) $m \equiv 11 \bmod 24$ :

$$\sum [E(m - 48\nu^2) - E(m - 3\nu^2)] = 0.$$

(8) $m \equiv 5 \bmod 6$ :

$$\sum (-1)^{\nu} E(m - 3\nu^2) = 0.$$

(9) $m \equiv 23 \bmod 24$ :

$$\sum [E(m - 12\mu^2) - E(m - 3\mu^2)] = 0.$$

(10) $m \equiv 3,\ 19 \bmod 24$ :

$$4 \sum [E(m - 48\nu^2) - E(m - 3\mu^2)] = A(m).$$

(11) $m \equiv 1,\ 3 \bmod 6$ :

$$8 \sum (-1)^{\nu} E(m - 3\nu^2) = C(4m).$$

(12) $m \equiv 7,\ 15 \bmod 24$ :

$$4 \sum [E(m - 12\mu^2) - E(m - 3\mu^2)] = -B(m).$$

(13) $m \equiv 1 \bmod 24$ :

$$12 \sum \left[ E\left(\frac{m - \mu^2}{12}\right) - E\left(\frac{m - 4\nu^2}{3}\right) \right] = H(m),$$

$$\mu \equiv 1, 5 \bmod 6, \qquad \nu \equiv 2, 4 \bmod 6;$$

namely, the $\sum$ refers only to such odd $\mu \gtrless 0$, and to such arbitrary $\nu \gtrless 0$, as indicated, and similarly in (14), (17).

(14) $m \equiv 13 \bmod 24$ :

$$12 \sum \left[ E\left(\frac{m - \mu^2}{12}\right) - E\left(\frac{m - 4\nu^2}{3}\right) \right] = -H(m),$$

$$\mu \equiv 1,\ 5 \bmod 6, \qquad \nu \equiv 1,\ 5 \bmod 6.$$

(15) $m \equiv 3 \bmod 8$ :

$$12 \sum \left[ E\left(\frac{m - 3\mu^2}{4}\right) - E(m - 48\nu^2) \right] = H(3m).$$

(16) $m \equiv 7 \bmod 8$ :

$$12 \sum \left[ E\left(\frac{m - 3\mu^2}{4}\right) - E(m - 12\mu^2) \right] = - H(3m).$$

(17) $m \equiv 1 \bmod 6$ :

$$24 \sum \left[ E\left(\frac{m - \mu^2}{3}\right) - E\left(\frac{m - 4\nu^2}{3}\right) \right] = K(4m),$$

$$\mu \equiv 1, 5 \bmod 6, \qquad \nu \equiv 1, 2 \bmod 3.$$

(18) $m \equiv 7 \bmod 8$ :

$$24 \sum [E(m - 3\mu^2) - E(m - 12\mu^2)] = K(12m).$$

4. *Proofs.* As remarked in §1, it is unnecessary to go into details, although some of the formulas in §§2–3 follow only after several reductions. It is essential however to state from what identities of Gauss all of the preceding follow as indicated. The requisite relations are given (in a slightly different notation) by Gauss (*Werke*, vol. 3, p. 471), and are

$$3Pp = \frac{r^3}{R} - \frac{q^3}{Q}, \qquad Pp = \frac{Q^3}{q} - \frac{R^3}{r}$$

where, for $|x| < 1$,

$$p \equiv \sum x^{\nu^2}, \quad q \equiv \sum (-1)^{\nu} x^{\nu^2}, \quad r \equiv \sum x^{\mu^2/4},$$

the $\sum$ extending to all integers $\nu \gtrless 0$, or to all odd integers $\mu \gtrless 0$, and $P$, $Q$, $R$ are obtained from $p$, $q$, $r$ respectively on replacing $x$ by $x^3$. Independent proofs of Gauss' identities from the theta formula of Schröter have been given by Goring.*

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*Mathematische Annalen, vol. 7 (1874), pp. 325 et seq.

# HOMOGRAPHIC CIRCLES OR CLOCKS*

BY LULU HOFMANN AND EDWARD KASNER

1. *Introduction.* This paper contains some theorems on the homographic transformation of one circle into another. The distribution of points on the transformed circle is studied under the assumption that the distribution on the original circle is uniform. The varying *density* on the transformed circle is characterised *im Grossen* by the *centroid*, a point defined analytically by a mean-value process. At an individual point it is measured by the absolute value of the ratio of corresponding arc elements on the two circles, the ratio taken from the original to the transformed element.

The distribution of points on the transformed circle proves to be uniform when and only when the centroid coincides with the center. When this coincidence does not occur, the density varies as follows.

It reaches its maximum and its minimum, two reciprocal values, in the end points of the diameter through the centroid, the *main diameter*.

It assumes every intermediate value twice and to every value occurring the reciprocal value likewise occurs.

Points of equal density lie symmetrically with regard to the main diameter, and points of reciprocal densities are collinear with the centroid.

If we draw a chord through the centroid and number the end points and the corresponding segments determined by the centroid arbitrarily as the first and second, then the density in the first point will be equal to the ratio of the length of the second segment to that of the first segment.

2. *Homographic Clocks. The Centroid.* We shall treat the problem analytically. As the most convenient tool we choose the linear fractional transformation with complex coefficients of one complex variable into another

*Presented to the Society, February 6, 1928.

$$z = \frac{A + Bt}{C + Dt} \equiv \Omega(t).$$

This transforms the unit circle $T$,

$$t = e^{i\theta}, \qquad (\theta \text{ real}),$$

into the circle $Z$

$$z = \frac{A + Be^{i\theta}}{C + De^{i\theta}} \equiv w(\theta),$$

and these are the two circles we shall study.* They are homographically related on account of the linearity of the transformation formula. Geometrically the transformation is determined by three pairs of corresponding points on the two circles, because, since we are dealing with a homography, the anharmonic ratio of four points on $T$ is equal to that of the four corresponding points on $Z$. A circle whose points are in definite homographic correspondence with the points of the unit circle, distinguished points we term a *homographic clock*; and more precisely a *positive* or *negative homographic clock* according as when $t$ moves on the circle $T$ in a certain sense, $z$ moves on the circle $Z$ in the same or in the opposite sense (the sense is uniquely determined on account of the one-to-one character of the transformation).

The concept of *homographic clock* belongs to both *inversive* and *projective* geometry.

With regard to the unit circle $T$, we note that $t = -C/D$ and $t = -\overline{D}/\overline{C}$ are corresponding points in inversion. The same relation must therefore hold with regard to the circle $Z$ for the transformed points

$$z = \Omega\left(-\frac{C}{D}\right) = \infty, \quad z = \Omega\left(-\frac{\overline{D}}{\overline{C}}\right) = \frac{A\overline{C} - B\overline{D}}{C\overline{C} - D\overline{D}}.$$

And since the point at infinity is the conjugate of the center of the circle of inversion, it follows that $Z$ has the center

$$N = \frac{A\overline{C} - B\overline{D}}{C\overline{C} - D\overline{D}}.$$

*When in the following $z$ occurs and nothing is expressly remarked to the contrary, then $z = w(\theta)$ is always meant.

Furthermore, forming $|N-w(0)|$ (we could also have chosen any other definite value for $\theta$) we find that $Z$ has the radius,

$$R = \left|\frac{AD-BC}{C\bar{C}-D\bar{D}}\right|.$$

The transformed circle $Z$ will therefore degenerate into a line when and only when $|C|=|D|$, which we henceforth exclude. To obtain a general idea of the distribution of points on the new circle, we form the mean value $M$ of $z$ with regard to $\theta$

$$M = \frac{1}{2\pi}\int_0^{2\pi} \frac{A+Be^{i\theta}}{C+De^{i\theta}}\,d\theta.$$

The point $M$ indicates the *mean point* into which the points of $T$ are transformed. We call it the *centroid* of the clock. For $D=0$ and $C=0$ the formulas simplify; we have

$$D=0:\quad z=\frac{A}{C}+\frac{B}{C}e^{i\theta},\quad N=\frac{A}{C},\quad M=\frac{A}{C};$$

$$C=0:\quad z=\frac{A}{D}e^{-i\theta}+\frac{B}{D},\quad N=\frac{B}{D},\quad M=\frac{B}{D}.$$

When neither $D=0$ nor $C=0$, the integrand becomes

$$\frac{B}{D}+\frac{AD-BC}{D}\cdot\frac{1}{C+De^{i\theta}},$$

and

$$M=\frac{1}{2\pi}\left[\frac{B}{D}\theta+\frac{1}{i}\left(\frac{A}{C}-\frac{B}{D}\right)\left(i\theta-\log\{C+De^{i\theta}\}\right)\right]_0^{2\pi}$$

$$=\frac{A}{C}-\frac{1}{2\pi i}\left(\frac{A}{C}-\frac{B}{D}\right)\epsilon,$$

where $\epsilon=0$ when $|C|>|D|$ and $\epsilon=2\pi i$ when $|C|<|D|$. Including the cases $D=0$ and $C=0$, we therefore have the following result.

*The centroid $M$ of the clock $Z$ lies at the point*

$$M = \frac{A}{C}, \quad |C| > |D|,$$

$$M = \frac{B}{D}, \quad |C| < |D|.$$

As the formulas show, the center and the centroid coincide when $D=0$ or $C=0$. Vice versa, these are the only cases for which this coincidence occurs. For according to the two values of $M$ we have

$$N - M = \frac{A\overline{C} - B\overline{D}}{C\overline{C} - D\overline{D}} - \frac{A}{C} = \frac{\overline{D}(AD - BC)}{C(C\overline{C} - D\overline{D})},$$

$$N - M = \frac{A\overline{C} - B\overline{D}}{C\overline{C} - D\overline{D}} - \frac{B}{D} = \frac{\overline{C}(AD - BC)}{D(C\overline{C} - D\overline{D})},$$

and these expressions vanish when and only when $D=0$ or $C=0$, because $AD-BC=0$ would make $z$ a constant independent of $\theta$.

Among all homographic clocks, those of the types $z=A+Be^{i\theta}$ and $z=Ae^{-i\theta}+B$ corresponding to $D=0$ and $C=0$ are thus distinguished by the coincidence of $N$ and $M$. It follows immediately from the formulas that when the point $t$ moves with uniform velocity on the circle $T$, the point $z$ moves with uniform velocity on the circle $Z$, in the same sense as $t$ in the first case and in the opposite sense in the second case. We therefore term these clocks *uniform positive* and *negative clocks*.

*The uniform positive and negative clocks*

$$z = A + Be^{i\theta}, \quad z = Ae^{-i\theta} + B$$

*are the only homographic clocks for which the center and the centroid coincide.*

From the preceding discussion, by reason of continuity, we may deduce the following theorem,

*Any homographic clock*

$$z = \frac{A + Be^{i\theta}}{C + De^{i\theta}}$$

*is a positive or negative clock according as* $|C| > |D|$ *or* $|C| < |D|$.

3. *The Density.* After having thus studied the distribution of points on $Z$ *im Grossen* by constructing the mean point or centroid $M$, we now proceed to study it in the infinitely small, that is, at the individual points of $Z$. It will be represented by a quantity which for obvious reasons we call the *density* and abbreviate by $\delta$. The density will be characterised by $|dz/d\theta|$, the ratio of the corresponding arc lengths on the circles $Z$ and $T$. For example, let us consider the circle $T$ as uniformly covered with mass in such a manner that the mass on $d\theta$ is measured by $|d\theta|$. Then, since under the transformation which transforms $d\theta$ into $dz$, the mass may be considered constant, the density $\delta$ with which it is distributed on $dz$ must be such that $|dz| \cdot \delta = |d\theta|$, that is, the density will be the reciprocal of $|dz/d\theta|$.

The formulas simplify, if we do not study $|dz/d\theta|$ itself but the related expression $|dz'/d\theta|$, where

$$z' = (z - N)\frac{D\overline{D} - C\overline{C}}{AD - BC} = \frac{\overline{D} + \overline{C}e^{i\theta}}{C + De^{i\theta}}.$$

The circle $Z'$ is obtained from $Z$ by three simple operations in the $z$-plane: the translation represented by the vector $-N$ which carries the center of $Z$ into the origin; the rotation through the argument of $(D\overline{D} - C\overline{C})/(AD - BC)$, and the magnification from the origin with the factor $1/R$. Hence it is obvious that the distribution of points on $Z'$ is essentially the same as on $Z$ (namely equivalent by similitude). We therefore omit the primes and continue with the circle

$$z = \frac{\overline{D} + \overline{C}e^{i\theta}}{C + De^{i\theta}} \equiv w(\theta)$$

which we now call $Z$. It is a circle of unit radius with its

center at the origin. To fix on a definite case, let us assume $|C|>|D|$, so that

$$M=\frac{\overline{D}}{C}.$$

It will be seen immediately from the results obtained that they hold equally for the other case.

By differentiation we obtain

$$\left|\frac{dz}{d\theta}\right|=\left|\frac{ie^{i\theta}(C\overline{C}-D\overline{D})}{(C+De^{i\theta})^2}\right|=\frac{C\overline{C}-D\overline{D}}{C\overline{C}+D\overline{D}+\overline{C}De^{i\theta}+C\overline{D}e^{-i\theta}}.$$

Putting

$$|C|=c,\quad |D|=d,\quad \overline{C}D=cde^{ik},$$

we obtain

$$\left|\frac{dz}{d\theta}\right|=\frac{c^2-d^2}{c^2+d^2+2cd\cos(\theta+k)}\equiv\frac{1}{\delta(\theta)}.$$

*The density on the transformed circle Z is independent of $\theta$ when and only when $D=0$ or $C=0$, that is when the center and the centroid coincide.* This means that the clock is uniform.

By substituting $\theta'=\theta+k$ we obtain

$$z=\frac{\overline{C}d}{Dc}\cdot\frac{d+ce^{i\theta'}}{c+de^{i\theta'}}=w'(\theta'),\quad \left|\frac{dz}{d\theta}\right|=\frac{c^2-d^2}{c^2+d^2+2cd\cos\theta'}\equiv\frac{1}{\delta'(\theta')}.$$

It is obvious that the points $z=w'(\theta')$ and $z=w'(-\theta')$ lie symmetrically with regard to the diameter through the point $z=\overline{C}d/(Dc)$; and since $\delta'(\theta')=\delta'(-\theta')$, the density at these points is the same. (Vice versa any two points of equal density have the above described mutual position.) At the point $z=\overline{C}d/(Dc)$ itself, $\theta'=0$, and $\delta$ reaches its maximum

$$\delta_{max}=\frac{c+d}{c-d}.$$

At the diametrically opposed point $z=-\overline{C}d/(Dc)$, $\theta'=\pi$ and $\delta$ assumes its minimum

$$\delta_{min} = \frac{c-d}{c+d} = \frac{1}{\delta_{max}}.$$

Referring to the maximum and minimum value of the density, we denote these points by $z_{max}$ and $z_{min}$:

$$z_{max} = \frac{\overline{C}d}{Dc}, \quad z_{min} = -\frac{\overline{C}d}{Dc}.$$

The diameter through $z_{max}$ and $z_{min}$ we term *the main diameter.*

*The distribution of points on the circle Z is symmetric with regard to a definite diameter, the main diameter. The density is equal in points lying symmetrically with regard to this diameter and reaches its maximum and minimum, which are reciprocal, in its end points.*

We have

$$\frac{z_{max}}{M} = \frac{c}{d}, \quad \frac{z_{min}}{M} = -\frac{c}{d},$$

$$\left|\frac{z_{max}-M}{z_{min}-M}\right| = \frac{c-d}{c+d} = \delta_{min} = \frac{1}{\delta_{max}}.$$

This means:

*The centroid lies on the main diameter and divides it in such a manner that the ratio of its distance from the first end point to its distance from the second end point is equal to the density at the second point.*

It is obvious, from the fact that the maximum and minimum density are reciprocal, that to every value of the density occurring the reciprocal value must also occur. We shall now show that for points with reciprocal densities a theorem holds which is the immediate extension of the one just stated.

Denote by $z=w(\theta)$ and $z^*=w(\theta^*)$ any two points collinear with the centroid and by $l$ and $l^*$ their respective distances from this point. We have

$$l = \left| \frac{\bar{D} + \bar{C}e^{i\theta}}{C + De^{i\theta}} - \frac{\bar{D}}{C} \right| = \left| \frac{(c^2 - d^2)e^{i\theta}}{C(C + De^{i\theta})} \right|.$$

Without calculating $\theta^*$ itself, one easily finds

$$l^* = \left| \frac{c^2 + d^2 + 2cd \cos(\theta + k)}{C(C + De^{i\theta})} \right|,$$

so that

$$\frac{l}{l^*} = \frac{c^2 - d^2}{c^2 + d^2 + 2cd \cos(\theta + k)} = \frac{1}{\delta(\theta)}.$$

But similarly of course

$$\frac{l^*}{l} = \frac{1}{\delta(\theta^*)}.$$

Therefore

$$\delta(\theta) = \frac{1}{\delta(\theta^*)}.$$

*On the transformed circle Z the density is reciprocal in two points z and $z^*$ which are collinear with the centroid. The density at the point $z(z^*)$ is equal to the ratio of the distances of $z^*(z)$ and $z(z^*)$ from the centroid.*

## Appendix on Polygenic Functions

BY EDWARD KASNER

The geometric concept of a *clock* first presented itself in the theory of *polygenic functions*†

$$w = \phi(x,y) + i\psi(x,y),$$

as follows. The derivative

$$\frac{dw}{dz} = L + le^{-2i\theta},$$

where

$$L = \mathfrak{D}w = \tfrac{1}{2}(D_x - iD_y)w,$$

$$l = \mathfrak{P}w = \tfrac{1}{2}(D_x + iD_y)w.$$

† See E. Kasner, Science, vol. 66 (1927), pp. 581–582, and Proceedings of the National Academy of Sciences, vol. 14 (1928), pp. 75–82.

This gives for each point $x+iy$ a uniform negative clock with center vector $L$ and phase vector $l$.

It is to be noticed that the angle $\theta$ of the preceding joint paper is here $2\theta$ because we are dealing with a correspondence between the points of the derivative circle and a pencil of directions, rather than with a unit circle.*

If $w_1$ and $w_2$ are any two polygenic functions, then the derivative is

$$\frac{dw_1}{dw_2} = \frac{L_1 + l_1 e^{-2i\theta}}{L_2 + l_2 e^{-2i\theta}},$$

and so defines a general homographic clock. In particular, the clock degenerates into a point when and only when the two functions are of the same class.† The clock becomes a straight line (the case omitted in the joint paper) when and only when the jacobian of $\phi_2$ and $\psi_2$ vanishes. Finally, the clock becomes uniform (negative or positive) when and only when the denominator function $w_2$ is an analytic function of $x+iy$ or $x-iy$. Hence the mean value (centroid)

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{dw_1}{dw_2} d\theta$$

coincides with the center of the derivative clock when and only when $w_2$ is an analytic function of $x+iy$ or $x-iy$. The theory of congruences of clocks will be studied elsewhere.

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* See the abstracts in this Bulletin, vol. 34 (1928), 152, pp. 263–264.

† According to the terminology introduced by Hedrick, Ingold, and Westfall, *Theory of non-analytic functions of a complex variable*, Journal de Mathématiques, (9), vol. 2 (1923), 327–342.

# CONCERNING ACCESSIBILITY IN THE PLANE AND REGULAR ACCESSIBILITY IN $n$ DIMENSIONS*

BY G. T. WHYBURN

1. *Accessibility in the Plane.* The point set $K$ is said to be everywhere accessible from a point set $R$ provided that if $A$ and $B$ are any two points belonging to $K$ and $R$ respectively, then there exists a simple continuous arc $AB$ from $A$ to $B$ such that $AB-A$ is a subset of $R$. In my paper *Concerning plane closed point sets which are accessible from certain subsets of their complements*,† among other results, I proved the following theorem.

THEOREM A. *If, in a plane $S$, $K$ is a point set such that there exist three mutually exclusive connected subsets $R_1$, $R_2$, and $R_3$ of $S-K$ such that $K$ is everywhere accessible from $R_1$ and $R_2$ and every point of $K$ is a limit point of $R_3$, then $K$ contains not more than two points.*

In this section it is purposed to establish some results which are related to Theorem A and in the proof of which use will be made of Theorem A. It is obvious from Theorem A that there does not exist any plane continuum, which is everywhere accessible from each one of a particular group of three of its complementary domains. It is not so apparent, however, that there does not exist a plane continuum each point of which is accessible from each one of some group of three of the complementary domains of this continuum. This result will be demonstrated in this present paper as an immediate consequence of a somewhat more general theorem. It will also be shown that if each point of a bounded plane continuum $M$ is accessible from each of two complementary

* Presented to the Society, December 28, 1927.

† Offered to the Proceedings of the National Academy of Sciences. See Theorem 1, and remark at the end of proof of Theorem 1.

domains of $M$, then $M$ is a Menger regular curve* which has at most a countable number of cut points.

THEOREM 1. *If $G$ is any countable collection of mutually exclusive connected point sets in the plane $S$, $T$ denotes the sum of all the point sets of the collection $G$, and $K$ denotes the set of all points $X$ in $S-T$ such that $X$ is accessible from at least three of the sets of $G$, then $K$ is countable.*

PROOF.† Suppose, on the contrary, that $K$ is uncountable. Then since $G$ is countable and each point of $K$ is accessible from at least one set of the collection $G$, it follows that there exists a set $R_1$ of the collection $G$ such that each point of an uncountable subset $K_1$ of $K$ is accessible from $R_1$. Since $K_1$ is uncountable and the collection $G_1$ obtained by omitting $R_1$ from the collection $G$ is countable, and since each point of $K$ is accessible from at least one set of $G_1$, it follows that there exists a set $R_2$ of the collection $G_1$ such that each point of an uncountable subset $K_2$ of $K_1$ is accessible from $R_2$. Let $G_2$ be the collection obtained by omitting the set $R_2$ from the collection $G_1$. Then since $K_2$ is uncountable and $G_2$ is countable, and each point of $K_2$ is accessible from at least one set of the collection $G_2$, there exists a set $R_3$ of the collection $G_2$ such that each point of an uncountable subset $K_3$ of $K_2$ is accessible from $R_3$. But then every point of $K_3$ is accessible from each of the three mutually exclusive connected subsets $R_1$, $R_2$, and $R_3$ of $S-K_3$, and hence, by Theorem A, $K_3$ can contain at most two points. But $K_3$ is uncountable. Thus the supposition that $K$ is uncountable leads to a contradiction; and the truth of Theorem 1 is therefore established.

Since the complementary domains of every plane continu-

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* See the definition of this term below.

† Professor W. A. Wilson has kindly called my attention to the following very concise proof of Theorem 1: Since $G$ is countable, the number of groups of three sets of $G$ is countable. It follows from Theorem A that at most two points of $K$ are accessible from each set of any one of these groups. Hence the set of points of $K$ must be countable.

um are countable, we have the following immediate corollaries.

COROLLARY 1. *If $M$ is any plane continuum, then not more than a countable number of points of $M$ are accessible from each of three complementary domains of $M$.*

COROLLARY 2. *There does not exist, in the plane, a continuum $M$ every point of which is accessible from each of three distinct complementary domains of $M$.*

DEFINITION. A continuum $M$ is said to be a Menger regular curve* provided that if $P$ is any point of $M$ and $\epsilon$ is any positive number, then there exists a connected open subset $R$ of $M$ which contains $P$, is of diameter $<\epsilon$, and whose $M$-boundary is finite. The $M$-boundary of an open subset $R$ of a point set $M$ is the set of all those points of $M-R$ which are limit points of $R$.

THEOREM 2. *If every point of a plane continuum $M$ is accessible from at least two of the complementary domains of $M$ then $M$ is a Menger regular curve; and if $M$ is bounded, then it has not more than a countable number of cut points.*

PROOF. I shall first show that $M$ is a Menger regular curve. Suppose, on the contrary, that $M$ is not a Menger regular curve. Then by a theorem of Menger's† there exists a subcontinuum $H$ of $M$ no point of which is a regular point of $M$. Now let $D_1, D_2, D_3, \cdots$ be the complementary domains of $M$. For each $i$, let $B_i$ denote the boundary of $D_i$, and let $A_i$ denote the set of points $H \cdot B_i$. Then, for each $i$, $A_i$ is a closed point set. Since, by hypothesis, every point of $H$ is accessible from one of the domains $D_1, D_2, \cdots$, it follows that $H=A_1+A_2+A_3+\cdots$. It is well known that no continuum is the sum of a countable number of closed point sets each of which is totally disconnected. Therefore, for some integer $p$, $A_p$ contains a continuum $K$. Now for each integer $i \neq p$, let $C_i$ denote the set of points $K \cdot B_i$. Then

* See K. Menger, *Grundzüge einer Theorie der Kurven*, Mathematische Annalen, vol. 95 (1925), pp. 277–306.

† See K. Menger, loc. cit., p. 288, Theorem 8.

since, by hypothesis, every point of $K$ is accessible from at least one of the domains $D_1, D_2, \cdots, D_{p-1}, D_{p+1}, \cdots$, it follows that $K = C_1 + C_2 + \cdots + C_{p-1} + C_{p+1} + \cdots$. And since, for each $i$, $C_i$ is closed, it follows just as above that there exists an integer $q \neq p$ such that $C_q$ contains a bounded continuum $T$. The continuum $T$ belongs to the boundary of each of the domains $D_p$ and $D_q$.

Now all, save possibly a countable number, of the points of $T$ must be accessible from $D_p$. For if on the contrary, an uncountable subset $Q$ of $T$ exists, no point of which is accessible from $D_p$, then since, by hypothesis, every point of $T$ is accessible from at least two complementary domains of $M$, it follows by an argument similar to that given in the proof of Theorem 1 that there exists an uncountable subset $W$ of $Q$ and two integers $j$ and $k$ each distinct from $p$, and such that each point of $W$ is accessible from each of the domains $D_j$ and $D_k$. But since every point of $W$ is a limit point of $D_p$, this contradicts Theorem A. Hence all, save possibly a countable number, of the points of $T$ are accessible from $D_p$. Likewise all, save possibly a countable number of the points of $T$, are accessible from $D_q$. Therefore there exists a countable, or vacuous, subset $U$ of $T$ such that every point of $T - U$ is accessible from each of the domains $D_p$ and $D_q$.

Let $X$ and $Y$ be two points of $T - U$. Then since $X$ and $Y$ are accessible from each of the domains $D_p$ and $D_q$, it readily follows that there exist points $P$ and $Q$ in $D_p$ and $D_q$ respectively and arcs $PXQ$ and $PYQ$ from $P$ to $Q$ which have in common only the points $P$ and $Q$, and such that the point set $PX + PY - (X + Y)$ is a subset of $D_p$ and the set $QX + QY - (X + Y)$ is a subset of $D_q$. Let $J$ denote the simple closed curve $PXQYP$. Then since $T$ is bounded and has only the points $X$ and $Y$ in common with $J$, it follows by a theorem of Rosenthal's* that either the interior or the exterior of $J$

* A. Rosenthal, *Teilung der Ebene durch irreduzible Kontinua*, Sitzungsberichte der Mathematisch-Physikalischen Klasse der Bayerischen Akademie der Wissenschaften zu München, 1919, p. 104.

contains a connected subset $g$ of $T$ such that each of the points $X$ and $Y$ is a limit point of $g$. The two cases are practically alike, so let us suppose that $g$ is a subset of the interior of $J$. Let $G$ denote the subset of $M$ which lies within $J$, and let $N$ denote the set of points $G+X+Y$. Then since $g+X+Y$ is connected it follows that $N$ is a continuum.

Let $E$ denote the set of all those points of $T-U$ which belong to $g$. Then since $U$ is countable, and $g$ is a connected subset of $T$, it follows that $E$ is uncountable. Let $Z$ be any point of $E$. Then since $Z$ is accessible from each of the domains $D_p$ and $D_q$, it readily follows that there exists an arc $AZB$, where $A$ is a point on the arc $XPY$ of $J$ and $B$ a point on the arc $XQY$ of $J$, and such that $AZ-Z$ is a subset of $D_p$ and $BZ-Z$ is a subset $D_q$. The arc $AZB$ divides the interior of $J$ into two regions $R_x$ and $R_y$ whose boundaries contain $X$ and $Y$ respectively. And if $S_x$ denotes the set of points $N \cdot R_x+X$ and $S_y$ the set of points $N \cdot R_y+Y$, then it is clear that $S_x$ and $S_y$ are mutually separated sets, and that $S_x+S_y=N-Z$. Therefore each point $Z$ of $E$ is a cut point of $N$. But since $E$ is uncountable, then by a theorem of the author* it follows that $E$ contains at least one point $V$ which is a regular point (in fact a point of Menger order two) of $N$; and since $M-N$ contains no point within $J$, it follows that $V$ is a regular point of $M$. But $V$ belongs to $H$, and by supposition no point of $H$ is a regular point of $M$. Thus the supposition that $M$ is not a Menger regular curve leads to a contradiction.

Now suppose $M$ is bounded, and that, contrary to the second part of this theorem, it has uncountably many cut points. Then by the above quoted theorem of the author, at least one of these cut points of $M$ must be a point of Menger order two of $M$. But by another theorem of the author† a cut point of Menger order two of a bounded

* G. T. Whyburn, *Concerning the cut points of continua*, Transactions of this Society, vol. 30 (1928), No. 3. The theorem used here states that *all, save possibly a countable number, of the cut points of a plane continuum M are points of Menger order two of that continuum.*

† Loc. cit., Theorem 10.

continuum $M$ can belong to the boundary of at most one complementary domain of $M$; and by our hypothesis every point of $M$ must belong to the boundaries of at least two complementary domains of $M$. Thus the supposition that $M$ has uncountably many cut points leads to a contradiction, and the theorem is proved.

It is of interest to note that every bounded continuum satisfying the hypothesis of Theorem 2 is a type of continuous curve which has elsewhere* been called a *two-way continuous curve*, that is, every two of its points can be joined by two different arcs which lie wholly in the curve.

2. *Regular Accessibility in n Dimensions.* A limit point $P$ of a point set $R$ is said to be regularly accessible† from $R$ provided that if $\epsilon$ is any positive number, then there exists a $\delta_{\epsilon p}>0$ such that every point $X$ of $R$ whose distance from $P$ is $<\delta_{\epsilon p}$ can be joined to $P$ by an arc $XP$ of diameter $<\epsilon$ such that $XP-P$ is a subset of $R$.

THEOREM 3. *In order that the boundary point $P$ of a connected open subset‡ $R$ of a continuous curve $M$ in a euclidean space of $n$ dimensions should be regularly accessible from $R$, it is necessary and sufficient that the point set $R+P$ should be connected im kleinen at $P$.*

PROOF. That the condition is necessary is an immediate consequence of the definition of regular accessibility. I shall show that the condition is sufficient. The case where $P$ belongs to $R$ is evident, so let us suppose that $P$ belongs to $M-R$. Let $\epsilon$ be any number $>0$. By hypothesis there exists a $\delta_{\epsilon p}>0$ such that every point $X$ of $R$ at a distance $<\delta_{\epsilon p}$ from $P$ can be joined to $P$ by a connected subset of $R+P$ of diameter $<\epsilon/2$. Let $G$ denote the set of all those

* See G. T. Whyburn, *Two-way continuous curves*, this Bulletin, vol. 32 (1926), pp. 659–663.

† See G. T. Whyburn, *Concerning the open subsets of a plane continuous curve*, Proceedings of the National Academy, vol. 13 (1927), pp. 650–657.

‡ The subset $R$ of a closed point set $M$ is an open subset of $M$ provided that $M-R$ is either vacuous or closed.

points of $R$ whose distance from $P$ is $<\delta_{\epsilon p}$, let $X_1$ be any point of $G$, and let $R_1$ denote the maximal connected subset of the set of all those points of $R$ whose distance from $P$ is $<\epsilon/2$ which contains $X_1$. Then $R_1$ is a connected open subset of $M$ which is of diameter $<\epsilon$; furthermore, $P$ is a limit point of $R_1$. For suppose it is not. By hypothesis a connected subset $N$ of $R+P$ exists which contains both $P$ and $X_1$ and is of diameter $<\epsilon/2$. Since $N$ is connected and contains the point $X_1$ of $R_1$ and the point $P$ which is neither a point nor a limit point of $R_1$, it therefore contains a point $Q$, distinct from $P$, which belongs to $R-R_1$ and is a limit point of $R_1$. But $Q$ is at a distance $<\epsilon/2$ from $P$, and by definition $R_1$ is a maximal connected subset of the set of all points of $R$ a distance $<\epsilon/2$ from $P$. Thus the supposition that $P$ is not a limit point of $R_1$ leads to a contradiction. Hence $P$ is a limit point of $R_1$. In a similar way it follows that there exists a connected open subset $R_2$ of $M$ which is a subset of $R_1$, is of diameter $<\epsilon/2$, and has $P$ for a limit point; and so on. In general, for each positive integer $n$, a connected open subset $R_n$ of $M$ exists which is a subset of $R_{n-1}$, is of diameter $<\epsilon/n$, and has $P$ for a limit point. For each $n$, the set $R_n$ contains a point $X_n$ and an arc* $X_nX_{n+1}$. It is easy to see that the point set $P+X_1X_2+X_2X_3+\cdots+X_nX_{n+1}+\cdots$ is closed and that it contains an arc $X_1P$ of diameter $<\epsilon$ such that $X_1P-P$ is a subset of $R$. And since $X_1$ is any point of $G$, it follows that $P$ is regularly accessible from $R$.

COROLLARY. *In order that a boundary point $P$ of a domain $D$, in $n$-dimensional space, should be regularly accessible from $D$ it is necessary and sufficient that the set of points $D+P$ should be connected im kleinen.*

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* See R. L. Moore, Mathematische Zeitschrift, vol. 15 (1922), Theorem 1.

## BELL ON ALGEBRAIC ARITHMETIC

*Algebraic Arithmetic.* By E. T. Bell. Colloquium Publications of the American Mathematical Society, volume 7, 1927. 180 pp.

This book of marked originality is of vital interest to advanced students in various branches of mathematics, including the theory of numbers, abstract algebra, elliptic and theta functions, Bernoullian numbers and functions, and the foundations of mathematics.

A central feature is the new presentation of the author's principle of arithmetical paraphrases, which won him the Bôcher prize in 1924, jointly with Professor Lefschetz. This general principle serves to unify and extend many isolated results in the theory of numbers. In a modified form, this principle may be stated very simply for a special, but typical, case. Suppose that by two developments of an elliptic or theta function we have found a linear identity between the cosines of the angles $a_i x+b_i y$ for $i=1, 2, \cdots$; then if $f(a,b)$ is any function which is unaltered by the change of $a$ to $-a$ or of $b$ to $-b$, we have the same linear identity between the $f(a_i,b_i)$. Since we are free to choose the even function $f$, we obtain an infinitude of arithmetic facts.

The general principle of paraphrase (pp. 66–68) starts with a linear identity between terms each a product of $r$ cosines and $s$ sines and leads to the same identity between values of any function which is even in $r$ variables and odd in $s$. It holds also in the extended sense that the variables are any one-rowed matrices. Several illustrations are given to show in detail how the initial identities are obtained from theta functions of one or more arguments. The applications include many classic problems in the theory of numbers.

A leading feature of the book seems to the reviewer to be its success in a systematic attempt to find a unified theory for each of various classes of related important problems in the theory of numbers, including its interrelations with algebra and analysis. An older case was the development of a theory of abstract groups which includes as special cases the really vital results and processes of the various theories of concrete groups, such as groups of permutations of letters, groups of linear or other substitutions, and groups of motions. The gain is not merely in great economy, but in the clarification which results from the exclusion of the irrelevant and the concentration on the relevant. Here it was clearly necessary to proceed abstractly. To the immature mind, abstract means obscure. To anyone really initiated in modern mathematics, an abstract theory is the really clear one, since irrelevant facts have been discarded, and the vital facts and processes alone are retained. As between an abstract theory and a concrete one, the former is far more likely to be clear and sound, since its very nature requires that the assumptions be explicitly stated (as postulates) and that the deductions be made by logical processes and without lapse into intuition, consciously or not.

It is therefore wholly to its credit that Bell's book is so largely abstract. In this way he has secured the utmost generality and hence insured permanency to his work. There is also the following important immediate gain. After constructing abstract theories for various classes of problems, he was in a position to decide if those theories are distinct or are (abstractly) identical. In the latter case we have not merely the often astonishing conclusion that two quite different theories are really equivalent, but the great advantage of being able to apply in a particular case whichever of the two theories is best adapted to it.

The book is however not abstract for the sake of being abstract, but because it gives the essence of a vast array of concrete results traced to their true sources and easily deduced from these few sources. It is not to be overlooked that the author has a minute first-hand acquaintance with the vast array of concrete facts in the theory of numbers. It would require many volumes to expound them and the result would bewilder the reader. How much better to have a brief book which epitomizes all these facts under a few abstract theories. This original and scholarly book is an honor to American mathematics.

L. E. DICKSON

## FORSYTH ON CALCULUS OF VARIATIONS

*Calculus of Variations.* By A. R. Forsyth. Cambridge University Press, 1927. xxii+656 pp.

Weierstrass made three very important contributions to the theory of the calculus of variations. Earlier writers had deduced necessary conditions on a minimizing arc $y=y(x)$ $(x_1 \leqq x \leqq x_2)$ by comparing the value which such an arc gives to the integral to be minimized with the value given by neighboring arcs of the form $y=y(x)+\delta y(x)$, where $\delta y=a\eta(x)$. By the use of variations $\delta y$ of another type, Weierstrass deduced a new necessary condition. He also introduced the parametric representation of curves, $x=x(t)$, $y=y(t)$, into the theory. This was important from the standpoint of the older writers because it enables one to vary impartially either of the variables $x$ or $y$. The effort to do this had caused considerable difficulty and misunderstanding in the earlier theory of the non-parametric case. The parametric theory is even more valuable, however, because it removes the geometric restrictions on the form of curves which are imposed by the non-parametric representation $y=y(x)$, a removal which is necessary for the complete investigation of many geometric problems. Finally Weierstrass formulated clearly the problems which he studied, distinguished clearly between conditions which are necessary for a minimum and those which are sufficient, and devised an ingenious sufficiency proof which under certain circumstances established the minimizing property of an arc $y=y(x)$ as compared with all other neighboring arcs $y=y(x)+\delta y(x)$, irrespective of the form of the variation $\delta y$.

In the book here reviewed Professor Forsyth shows that he has been influenced by the first two of the contributions which have just been mentioned as being due to Weierstrass, the so-called Weierstrassian necessary

condition and the introduction of the parametric representation of curves. The method which he uses is that of expansion in series of powers of the variations of the functions defining the curves, the various necessary conditions being deduced by the study and transformation of the first and second variations of the integral to be minimized. The cases which he considers are problems in the plane for integrals whose integrands contain first, or first and second, derivatives of the functions defining the arc, in both non-parametric and parametric forms, the analogous problems in three-space, isoperimetric problems, double integrals containing first or second partial derivatives and isoperimetric problems involving double integrals, and triple integrals involving first partial derivatives.

In the first paragraph of his introduction Professor Forsyth says: "The range of Mathematical Analysis, usually known as the Calculus of Variations, deals with one of the earliest problems of ordinary experience. The requirement was, and is, to obtain the most profitable result from imperfectly postulated data; and the data may possibly be subjected to conditions, which likewise are imperfectly postulated. When data and conditions are expressed in analytical form, the necessary mathematical calculations can not be effected directly, because of some essential deficiency in the information." It seems to me that this rather vague type of language, which is also used in other places in the book, gives a very vague impression of the theory of the problems of the calculus of variations. The data of these problems can be postulated with accuracy, and the properties deduced for their solutions will in general characterize those solutions completely.

There is a strong tendency among modern writers to avoid as far as possible the elaborate transformations of the second variation which characterized earlier theories. Legendre's condition, which was originally deduced from the second variation, is in many cases an immediate consequence of the condition of Weierstrass which can be proved without using the second variation at all. Jacobi's condition is a consequence of the beautiful envelope theorem of Darboux, Zermelo, and Kneser, which is a generalization of the string property of the evolute of a curve, or it may be deduced from the second variation by a method which requires almost no manipulation.* For these reasons I always regret to see the elaborate transformation theory of the second variation given such a prominent place as Professor Forsyth has given it in his book.

In a paper read at the Toronto Congress, and which has recently been printed for publication in the Proceedings of the Congress, I showed that the result of the transformation which Clebsch devised for the second variation, in the very general case of the Lagrange problem, is obtained as an immediate consequence of a theorem of Weierstrass expressing the difference between the values of an integral on two neighboring curves as an integral of its $E$-function. So much effort has been spent on these

* See the references in the footnote, this Bulletin, vol. 32 (1926), p. 392; for the parametric case, see the Transactions of this Society, vol. 17 (1916), p. 195.

transformations that it seemed to me worth while to try to coordinate the transformation theory with other well known results in the calculus of variations. If the second variation must be transformed it is desirable to do it in some such way as this.

It has frequently surprised me to find how much effort the early writers on the calculus of variations spent on the theory of problems for which the integrand functions contain higher derivatives than the first. There are very few special problems of this sort in the literature. Furthermore the theory of such problems is a special case of the theory of the Lagrange problem which has a much wider application. Professor Forsyth discusses only relatively simple forms of the problem of Lagrange but he studies at length problems in the plane and in three-space for which second derivatives occur in the integrand. The theory of the second variation is of course very complicated for these problems. The parametric theory for such cases could doubtless be simplified by the introduction of invariants under parametric transformation in place of higher derivatives in the integrand, as has been done by Radon.*

The last two hundred pages of the book are devoted to double and triple integrals, with applications to minimal surfaces, and the usual necessary conditions are deduced from the first and second variations. The method of transforming the second variation of parametric problems is a modification due to Professor Forsyth of that used by Kobb. The two final chapters are devoted to double integrals whose integrands contain second derivatives, and to triple integrals, and the theory of the second variation for these cases is more completely treated than I have seen it done elsewhere.

G. A. BLISS

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* *Über das Minimum des Integrals* $\int_{s_0}^{s_1} f(x, y, \theta, \kappa)ds$, Wiener Sitzungsberichte, vol. 119 (1910), p. 1257.

## ELDERTON ON CURVES AND CORRELATION

*Frequency Curves and Correlation.* By W. Palin Elderton. Second edition. London, C. & E. Layton, 1927. viii+239 pp. Price 15 shillings net+6d postage.

In such a rapidly developing subject as statistics for an early book to celebrate its twenty-first birthday with an enlarged second edition is phenomenal. The substantial value of the first edition of this book has been enhanced by addition of new material, rearrangement of the old, and a more attractive appearance. The formidable-looking "Key to the Actuarial Terms and Symbols Used" which for the non-actuarial reader stood as a *cave canem* at the beginning of the book has been transferred to the end, and other such rearrangements have been made, rendering the book more readable by avoiding the diversion of attention from the context. There are 67 more pages than in the original edition. This increase is due to the growth in number of Pearson's types of curves from seven to twelve, to the addition of a chapter on partial correlation of the simplest type, and to a somewhat fuller treatment of many topics.

The chief distinctive feature of the book is still the full discussion of the Pearson frequency curves. Now it is true that this system of curves is strictly empirical and does not offer the possibilities, suggested by seminvariants, of incorporating into the curve-fitting process ideas of a general sort regarding properties of compound distributions. Nevertheless they are convenient. Moreover, the modern development of statistics has been so strongly influenced by Karl Pearson that anyone wishing to make his way through statistical literature needs a knowledge of these frequency curves for the same reason that one needs a knowledge of the classics. Such a knowledge may be gained by the abridged reading of the book suggested in an appendix. A considerable part of the book, given to the details of talculation, may be omitted by the casual reader but is indispensible to che curve-fitter.

In a chapter discussing the curves of Edgeworth and the Scandinavians the author concludes on the basis of goodness of fit in examples, and on account of the difficulties connected with series converging, if at all, at uncertain speed, that these systems lack the practical advantages of Pearson's.

The chapter on correlation is radically changed by addition of new material, some of the old material being removed to an appendix. The increased space on correlation is in accord with the increased importance of the subject. The Hardy summation method, which ought to be better known, is applied to the calculation of correlations as well as of univariate moments.

The treatment of correlation of characters not quantitatively measurable presents clearly the chief work of Pearson on the subject. However there ought to be a warning against overworking these methods. In several cases which have come to the reviewer's attention biserial and tetrachoric correlations have been computed with vast labor when all that was wanted was to decide whether there was any correlation at all. These functions, with

their dubious assumption of normal correlation and their large and hazy probable errors, are far less efficient for the purpose than the simpler method of contingency, which yields an immediate and correct interpretation in terms of probability. To take the classic example used in the text, the calculation of an exact value for the correlation between vaccination and recovery from smallpox might well be postponed until, by a much easier and more dependable process, it has been shown that this correlation is significantly greater than zero. So much use of correlation is heuristic that the point is important.

Inverse probability lurks in the book without ever coming into full view. The revolutionary work of R. A. Fisher has found no place in the new edition, so that Chapter X, on probable errors, and Chapter XII, on the correlation ratio and contingency, are partially obsolete. For example, page 187 is directly contradicted by Fisher's conclusion in Metron (vol. 5 (1925), No. 3, p. 96). Still more serious from the standpoint of this book are the modifications in the Pearson curve-fitting theory which Fisher has shown to be necessary.

The argument used on page 181 in deriving the formula for the correlation between sampling deviations in class frequencies is unconvincing; for why, in case the number found in a class falls short of expectation, must the corresponding excess in other classes be distributed in exact proportion to their sizes? This can certainly not be expected to happen in all or even in many cases; only by some process of averaging not fully defined can the idea be used. The correlation formula in question may however be reached in a purely algebraic manner by generalizing to $n$ dimensions the ordinary derivation of the standard deviation of a Bernoulli distribution.

On page 48 there is an allusion to some formulas not readily found, and on page 43 it is not clear why, when the criterion is infinite, a root of the quadratic must be infinite rather than zero. It is difficult to see the use or derivation of the functional equation on page 153, and it may possibly mislead the incautious. However the book is on the whole excellent. It supplies a considerable body of valuable information in compact form, and will probably always be the standard book of reference on the Pearson curves.

HAROLD HOTELLING

## CHARBONNIER ON EXTERIOR BALLISTICS

*Traité de Balistique Extérieure.* By P. Charbonnier. Volume II: *Problème Balistique Principal, Troisième Partie; Les Théories Balistiques.* Paris, Doin, Gauthier-Villars, 1927. 797 pp.

This volume follows Volume I, (of 637 pages, 1921) devoted to *Les Théorèmes Généraux de la Balistique*, and is to be succeeded by a third volume in rational ballistics concerning secondary ballistic problems. Volume I was divided into four books and this volume also is divided into four books numbered from five to eight as follows: Book V, Monomial Resistance; Book VI, Flat Fire; Book VII, Ballistic Series; Book VIII, The Calculation of Trajectories by Successive Arcs. From the Introduction to Volume I, one learns that three further volumes on ballistic science are planned, Volume IV, Experimental Exterior Ballistics, Volume V, History of Exterior Ballistics (with bibliography), Volume VI, Tables. The subject of interior ballistics, which differs considerably in its problems and methods, is not to be touched upon in this series.

The present undertaking is essentially a careful (save for numerous misprints) revision with amplification of the author's two volume work (*Balistique Extérieure Rationnelle*) of 1907. Not only is the subject treated throughout in greater detail, but additional carefully written chapters on newer methods have been introduced. In 1907, Charbonnier's work placed him as one of the conspicuous ballisticians of the world. Since that time he has been made a general in the French Army, and is here progressing with customary French lucidity upon the most ambitious text on ballistics ever undertaken. This should ensure respectful consideration from the few technical scientists seriously interested in ballistics.

Works on ballistics from the time of Bashforth, tend to have a curiously provincial air. They can hope to be of significance chiefly to artillery officers, who, however, seldom have the preparation, interest, or opportunity to study them. Most experimental work is kept in confidential army files. There is also a strong patriotic tendency toward nationalistic bias that army training and army contacts naturally accentuate. Most books on ballistics give the reader the impression that progress in the subject is practically confined to the compatriots or even to the friends of the author. This is noticeable even since the war. This book is no exception.

Ballistics is inherently a branch of engineering. Not only are there a distractingly large number of physical factors present in the problem, but the number of independent variables that are actually taken into account in firing is so large, and these variable factors are to a great extent so uncontrollable, that simple accurate formulas are universally regarded as out of the question. Rules based on the normal probability curve are used in actual field conditions, and the numerical tables upon which original computations are constructed are themselves obtained by statistical methods involving more or less arbitrary assumptions. The only methods used in

the past have been to relegate the less important variations to a secondary place (not to be ignored, but to be treated as linear differential perturbations) and to incorporate only the most important features into the fundamental differential equation for a trajectory. What this equation is, depends in part on the particular problem, but is in no case as simple today as it was before the World War. In the sense of "rational mechanics", there is no rational ballistics.

For ballistics, as for every other developing branch of engineering, there is a group of present-day problems involved on the one side in seeking to test present-day assumptions, and if possible to establish these on a rational basis, and on the other side in devising means for application of these accepted principles with an accuracy consistent with the data and yet with the greatest possible simplification concordant therewith.

An excellent survey of the history of ballistics, replete with illustrative comparisons, explicit data, and critical comments on physical assumptions, is given by the long classical work of C. Cranz. The only criticism is that despite supplementary notes in later editions, the work closes essentially with the period before the World War, which changed the perspective even on old problems.

This work of Charbonnier bids fair to be considerably more extensive than that of Cranz. Unfortunately it is subject to the same limitation that the author's point of view is that of 1914 and this is not relieved by the mere insertion of additional chapters on new methods. Unlike the work of Cranz, this book attempts to deal with what appears to be a non-existent subject. In open and deserved admiration of the secure position of rational mechanics (in which French writers from Laplace to Appell appear to surpass themselves) this army officer effectually divorces ballistics at this stage from its only logical bases, historical and physical. We meet, consequently, for at least the first seven hundred pages of this volume, a collection of mutually contradictory theories. These are given no historical justification and one cannot make comparisons among them since the author himself fails to compare their relative accuracy and withholds as yet all physical data from the reader in the effort to be strictly rational. It may be of interest to American readers that a condensed but just account (covering five pages) of American methods with reference to Major Moulton is found in the chapter, "Exposé de quelques méthodes de calcul."

Perhaps the most severe criticism of this book by Charbonnier (in view of its date of publication) is contained in his own words, à propos of the artificial solutions of Drach. "Les solutions, . . . , quelque intéressantes qu'elles soient, ne remplissent pas les deux conditions simultanées qu'exigent les balisticiens: 1° s'appliquer à une forme de la fonction $F(v)$, acceptable, c'est-à-dire suffisamment voisine de la fonction expérimentale de la résistance de l'air; 2° conduire, pour les autres éléments du mouvement à une solution formelle, susceptible d'être utilisée pour les applications numériques."

Possibly reference to future volumes of this projected series will serve to restore in part the needed perspective here so effectively eradicated.

A. A. BENNETT

## DOWSETT ON CONSTRUCTIVE GEOMETRY

*Advanced Constructive Geometry.* By J. F. Dowsett. Oxford University Press (American Branch, New York), 1927. viii+340 pp. Price $9.00.

The author of this work, who is lecturer and chief instructor in geometry and graphics, London County Council School of Building, Brixton, states in his preface that the book is intended for advanced students and for teachers of building geometry.

It is encouraging to know that such a thorough course in descriptive geometry, with applications to carpentry and masonry, is available for students of building geometry. Much too often, craftsmen have no adequate knowledge of the principles of descriptive geometry, and also no facilities for carrying out its constructions when they have need for them. Thus the author remarks (Chapter XVII, p. 278), "The absence of facilities for setting out elaborate geometrical constructions on the building site is the chief factor in perpetuating crude rule-of-thumb methods, and is responsible for much of the poor craftsmanship to be found in quite important structures." For this reason the author devotes a chapter to methods of placing lines to which to cut bevels, and for other purposes, directly on the materials of construction.

This is but one illustration of the practical nature of the work. However, aside from applications, the author gives the reader a very good working knowledge of Mongean Descriptive Geometry. In addition to this, the student will find much of interest and of value in the numerous applications to wood and stone cutting.

The book is well planned in that necessary material is developed and ready for use when needed. Thus, for example, the first six chapters, which constitute Section I, are devoted to constructions in plane geometry. Chapter I deals with some constructions (necessarily approximate) such as finding tangents and radii of curvature of curves which are given graphically (i.e., of curves for which the law of generation is not known). It also gives approximate constructions for rectifying and dividing arcs of circles. In the following chapters of this section much useful information is given concerning the circle and the other conic sections. Such applications are given as the construction of scroll curves and volutes, the drawing of moulds for easings to connect two straight hand-rails, etc.

The second section of the work, comprising Chapters VII to XX inclusive, is devoted to descriptive geometry, that is, to the solution of space problems by means of constructions which can be executed in the plane.

With the exception of Chapter XX, which deals with perspective, the method of treatment is principally that of double-orthographic projection (i.e., the method of Monge). Aside from the applications, the geometric problems treated are those usually considered in a thorough course on descriptive geometry.

No classification of problems, such as a division of them into problems of pure geometry of position, perpendicularity problems, and metrical problems, is given. Much, almost excessive, use is made of the device of "changing the ground line."*

Following the problems involving the point, line and plane, there is a consideration of the so-called "three round bodies": sphere, cylinder, cone, to each of which a chapter is devoted.

Among the applications to masonry in these chapters are methods of obtaining moulds (or templates) used in the construction of voussoirs for hemispherical domes, pendentives, cylindrical vaulting, groined arches, skew arches, etc. Equally important problems in carpentry are also considered in much detail.

A further chapter is devoted to what the author calls the geometry of double circular work. Here such problems as the design of arches over openings in cylindrical walls are considered for both stone and wood construction. A whole chapter is devoted to the geometry and construction of wreathed hand rails. Then follows the chapter, already mentioned, on the placing of lines directly on the materials of contruction, and another on mouldings and the design of moulding cutters. A chapter on shadows and one on perspective close the main body of the text.

Following the index is a well selected group of miscellaneous problems derived from the examination papers of various institutions.

The book is rather free from error. Most of the typographical errors are easily detected. In the statement of Problem 78, p. 145, concerning the determination of a point of a given plane for which the lines connecting this point with two given points of space shall make equal angles with the plane, it should be further stipulated that the plane of these two connecting lines be perpendicular to the given plane. For, as solved, the problem is nothing more than that of finding the brilliant point of a plane with respect to two given points of space.

In Fig. 126, the vertical trace of the plane containing the point $P$ is not in proper position. It should also be stated that this plane is parallel to the ground line.

The work in a number of places might have been made easier of understanding if a statement of the space construction (i.e., the operations in space of which the constructions in the drawing plane are counterparts) had preceded the description of the construction in the drawing plane.

The drawings are well executed and the pictures showing space relations (which the author calls pictorial views) are excellent. However, in several cases the scale is a little too small for the easy reading of the letters.

W. H. ROEVER

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* This consists in assuming a new vertical (or horizontal) plane of projection to which the space object under consideration is more simply related than it is to the old one, and then revolving (or "rebatting") this new plane into the horizontal (or vertical) plane around its horizontal (or vertical) trace, that is, around the new ground line.

## KLEIN ON NINETEENTH CENTURY MATHEMATICS

*Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert.* By Felix Klein. Teil I. Berlin, Springer, 1926. xiii+385 pp.

Twenty years ago, during a series of walks in the forest about Hahnenklee, in the Hartz Mountains, the conversation between Klein and a companion covered, as would naturally be the case, a wide range. Three statements, however, impressed his listener* very strongly. One was political: "There was a time when we looked up to England socially, politically, and as a naval power,—but that is a thing of the past." The second was political and of military significance: "America has no standing army today; twenty-five years hence she will have a large one." It is not strange that his auditor wondered at the real significance of these two statements by a man of Klein's vision and prominence. The third remark was in response to a statement to the effect that he of all men was the one to write a history of mathematics in the 19th century. "I am too old," was the reply, "It needs a young man who could devote years to its preparation." When it was urged that he had seen the development and had taken part in it as few if any others living had done, he remarked, "No, all that I could do would be to give a few lectures on the great events, but I am too much occupied to prepare even these." Ten years later, when the war was on, and his family had been sorely stricken, he gave these very lectures in his home in Göttingen, before a small group of listeners anxious to receive from a master that which only a master could give.

The lectures have been edited by Professors Courant and Neugebauer and are published as Band XXIV of Die Grundlehren der mathematischen Wissenschaften, a recent series, already well known to all mathematical students. Rightly did they say in their Vorwort: "Diese Vorlesungen sind die reife Frucht eines reichen Lebens inmitten der wissenschaftlichen Ereignisse, der Ausdruck überlegener Weisheit und tiefen historischen Sinnes, einer hohen menschlichen Kultur und einer meisterhaften Gestaltungskraft; sie werden sicherlich auf alle Mathematiker und Physiker und weit über diesen Kreis hinaus eine grosse Wirkung ausüben."

The work is divided into eight chapters. The first naturally begins with the founder of the modern German school of pure and applied mathematics, —Gauss. It considers his work in the applied field with respect to astronomy, geodesy, and physics, the last in connection with A. von Humboldt and Wilhelm Weber. In pure mathematics the attention is given chiefly to the number theory, forms, and the function theory, with a succinct statement as to the claim that Gauss is entitled to the award of priority

* The reader may suspect the identity of the "listener," who modestly conceals it. The editors may state at least that the conversation was actual and that this report is a first-hand report.

(THE EDITORS)

in the discovery of non-euclidean geometry, although not to priority of publication (pp. 57–60).

Chapter II relates to France and the École Polytechnique in the first decade of the century. In this Klein pays high tribute to the work of men like Fourier, Cauchy, Poncelet, Monge, and Galois. Of Cauchy he speaks as one "der sich mit seinen glänzenden Leistungen auf allen Gebieten der Mathematik fast neben Gauss stellen kann."

Chapter III concerns the founding of Crelle's Journal and the rise of pure mathematics in Germany. In this period the names mentioned as most prominent are those of Abel, Jacobi, Moebius, Plücker, and Steiner.

Chapter IV considers the development of algebraic geometry by Moebius, Plücker, and Steiner, with reference to Lagrange, Chasles, Cayley, Sylvester, Salmon, Beltrami, and Clifford, as well as to his own countrymen,—Riemann, Hesse, and Grassmann.

Chapter V deals with mechanics and mathematical physics in Germany and England before 1880,—the period of Hamilton, Thomson (Kelvin), and Maxwell in England, of Gibbs in America, and of Franz Neumann in Germany.

Chapter VI is devoted to the development of the theory of functions of a complex variable, chiefly at the hands of Riemann and Weierstrass, but with mention of the influence of Dirichlet, H. A. Schwarz, Fuchs, C. Neumann, Kovalevski, and others.

Chapter VII is taken up with the study of algebraic forms, and Chapter VIII with that of the theory of groups and automorphic functions, with special reference to the work of Galois, C. Jordan, Hermite, Riemann, and Poincaré.

It is proposed, in Band II, to treat chiefly of the theory of invariants and of relativity.

All who knew Professor Klein with any degree of intimacy are aware of his broadmindedness and catholicity of spirit. His love for science did not permit him to allow political prejudices to warp his judgment as to the tribute due to scholarship beyond the boundaries of his own country. It speaks well for his judicial spirit to observe that, of the scientists mentioned, nearly half were not of German nationality. Nearly a fourth were French, about an eighth were British, and a fair number were from Italy, Scandinavia, Russia, and our own country. It was natural that the work of the Germans should have been best known to him, and it was entirely fair that he should have ranked them higher than others of the last half of the century, not alone numerically but on the score of achievement.

DAVID EUGENE SMITH

## EVANS ON LOGARITHMIC POTENTIAL

*The Logarithmic Potential, Discontinuous Dirichlet and Neumann Problems.* By Griffith Conrad Evans. Colloquium Publications, American Mathematical Society, volume 6, 1927. viii+150 pp.

The appearance of this volume marks a broadening of the scope of the Colloquium Publications, in that these will no longer be confined to lectures given from time to time, by invitation of the American Mathematical Society, at its summer meetings. The new departure, however, could hardly have been initiated more smoothly or appropriately. For in the first place, Professor Evans has in the past been one of the Society's Colloquium lecturers, and secondly, his book is very closely related in character to the traditional Colloquium publications in that it is pre-eminently concerned with recent progress.

The book is characterized by the role played by Stieltjes and Lebesgue integrals and by functions of limited variation, and by the degree to which these concepts have enabled the author to express the conditions for his theorems in necessary and sufficient form. That these tools are particularly adapted to physical problems is a fact which the author early recognized. The material centers about a study of Poisson's integral in two dimensions and of the corresponding Stieltjes integral. The results are then extended to the integrals in terms of Green's functions for general regions.

In the introductory chapter, the fundamental properties of functions of limited variation and of Stieltjes integrals are succinctly developed, and some properties of Lebesgue integrals and theorems on the termwise integration of sequences are stated, with references or proofs. The last eight pages are given to Fourier series.

The second chapter develops properties of the Poisson-Stieltjes integral

$$u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1+r^2-2r\cos(\phi-\theta)}dF(\phi), \tag{A}$$

where $F(\phi)$ is of limited variation, and of the Poisson integral

$$u(r,\theta)=\frac{1}{2\pi}\int_0^{2\theta}\frac{1-r^2}{1+r^2-2r\cos(\phi-\theta)}f(\phi)d\phi, \tag{B}$$

where $f(\phi)$ is summable.

The function $u(r,\theta)$ defined by (A) is harmonic ($r<1$); the function

$$F(r,\theta)=\int_0^{\theta}u(r,\theta)d\theta$$

approaches, as $r\to 1$, $\theta$ being fixed, the value

$$\tfrac{1}{2}\{F(\theta+0)+F(\theta-0)\}-\tfrac{1}{2}\{F(0+)+F(0-)\};$$

$u(r,\theta)$ is the difference of two non-negative functions for $r<1$; $u(r,\theta)$ is a function of $\theta$ of limited variation, uniformly as to $r$; at every point $P(1,\phi)$ at which $F(\phi)$ has a derivative $F'(\phi)=f$ (i.e., almost everywhere), $u(r,\theta)\to f$

as $r \to 1$, under the sole restriction that $(r,\theta)$ remain between two chords of the unit circle terminating in $P$. These are among the properties derived. They are paralleled by those of the function $u(r,\theta)$ defined by (B).

The third chapter centers in the fundamental theorem, stated as follows in terms of the above function $F(r,\theta)$, and an infinite sequence

$$r_1, r_2, r_3, \cdots, r_n < 1, \lim r_n = 1.$$

(1) If $u(r,\theta)$ is harmonic for $r<1$, and if $F(r_n,\theta)$ is of limited variation as a function of $\theta$, uniformly as to $n$, then $u(r,\theta)$ admits the representation (A).

(2) If $u(r,\theta)$ is harmonic for $r<1$, and if $F(r_n,\theta)$ is absolutely continuous as a function of $\theta$, uniformly as to $n$, then $u(r,\theta)$ admits the representation (B).

This theorem, in combination with Chapter 2, gives necessary and sufficient conditions that a harmonic function admit the representation (A) or (B), as well as a considerable number of properties of such harmonic functions. A section is devoted to interesting special cases of functions given by (B). Of course the results of Chapter 2 include existence proofs of a very general character for the Dirichlet problem for the circle, and Chapter 3 then furnishes the desirable complements in the way of proofs of uniqueness. The formulation of these results constitutes the closing section of Chapter 3.

Chapter 4 is concerned with the potential of a simple distribution on the circumference of a circle of radius $a$:

$$\text{(C)} \qquad v(r,\theta) = -\frac{a}{2\pi}\int_0^{2\pi} \log\left[a^2 + r^2 - 2ar\cos(\phi - \theta)\right] dF(\phi) + A,$$

where $F(\phi)$ is periodic and of limited variation. A necessary and sufficient condition that $v(r,\theta)$, harmonic for $r<a$, admit a representation (C) is that

$$\int_0^{2\pi} \left| \frac{\partial}{\partial r} v(r,\theta) \right| d\theta$$

be bounded. Other properties of $v(r,\theta)$ follow, among them that

$$\lim_{r\to a} \int_0^{\vartheta} \frac{\partial v}{\partial r}\, d\theta = F(\theta) - F(0).$$

But this is in effect the statement that the flux of the vector field whose potential is $v$, across any given portion of the boundary, is a given quantity, and so we are led to a solution of the problem of Neumann having validity even when the normal derivative of $v$ does not exist on the boundary. The chapter is concluded by a study of the possibility of representing a function of the complex variable, analytic within a circle, by the Cauchy integral formula, the integral being extended over the circumference.

Chapter 5 contains the extension of results for the circle to the general simply connected region of the plane with more than one boundary point, and Chapter 6 to regions of finite connectivity. The theorems are stated in a form invariant under conformal transformation in terms of Green's

function and its conjugate, but careful reasoning is necessary to establish them. They are substantial generalizations, not merely obvious ones. Incidentally Chapter 6 contains some interesting results on isolated singularities of harmonic functions. The theorem ascribed to Lebesgue, by the way, on page 86, is due to Bôcher (see the reference on page 111).

The last chapter is a development of Professor Evans' symposium at the meeting of the Southwestern Section of the American Mathematical Society in November, 1926. At the beginning it reacts to simply connected regions, and establishes, among others, the result that the analytic transformation which carries the interior of the unit circle into the interior of a region with rectifiable boundary on a Riemann surface is conformal almost everywhere on the boundary. For regions of infinite connectivity a theorem is established which is concerned with the unique determination by its boundary values of a function harmonic and bounded in a region of infinite connectivity. The chapter closes with a discussion of convergence in the mean of various orders, relating a theorem of Noaillon with those of the author, and with a section on the characterization of harmonic functions by integral properties.

There are some thirty exercises at intervals throughout the text. These are designed in part to familiarize the reader with the concepts introduced, and in part to complement the theory, and they are to be taken seriously. This does not mean, however, that the book is to be regarded as a text, or a systematic introduction to potential theory. For satisfactory results, the reader should know something of potential theory, functions of a complex variable, and Lebesgue integrals. The author's hope, expressed in the preface, that the book will be found suitable for graduate students of a moderate degree of advancement will be fulfilled in the case of those with initiative: potential investigators. Others will have difficulty in supplying the details of reasoning suppressed by the author's succinct style. It is for investigators that the book will find its greatest usefulness, whether at the beginning of their careers or later.

In general, and in all essentials, the book is written with the care and thoroughness we should expect of its author. There are a few minor points where oversights have occurred. Thus Corollary 3 on page 7 is wrong, and the word uniformly has been omitted from the phrase "and (4) is convergent" in the last line of the second paragraph on page 126. These are the only cases noticed worth mentioning, and nothing is affected in the sequel.

The theory presented is Professor Evans' creation, most of the theorems developed being the results of his own researches. These here find a systematic and consecutive presentation. The arrangement is excellent. In spite of a well rounded character of the investigations set forth, the book leaves the reader with a sense of vistas opened. The methods and results cannot fail to influence further work on potential theory, Taylor's series, and elliptic differential equations. It will be indispensable to those occupied in these fields, and will hold an honored place in the colloquium publications.

O. D. KELLOGG

# SHORTER NOTICES

*Die Theorie der Gruppen von endlicher Ordnung mit Anwendungen auf algebräische Zahlen und Gleichungen sowie auf die Kristallographie.* By Andreas Speiser. Second edition, Berlin, Julius Springer, 1927. ix+254pp. Price 6.50 marks.

The first edition of this work appeared in 1923 and was reviewed in this Bulletin (vol. 29 (1923) p. 372). Hence it is only necessary to note here the improvements which the second edition presents. The fact that such an edition appeared so soon after the publication of the first tends to show that the work filled a real need. It is to be hoped that the new material which appears in the present edition will make the work still more useful and will tend to increase its popularity as an introduction to some of the most interesting and most modern developments of the large subjects to which it explicity relates.

The size of the volume has been increased by more than one-fourth of its original extent, while the number of different authors to whom reference is given has been more than doubled, being now 77. No additional name of an American author appears in this list, but the number of page references to the five which appeared in the original list has been more than doubled, while the total number of such references was increased relatively much less. One name which appeared in the original list does not appear in the present one, viz., that of E. Netto, who was credited in the original edition with a material simplification of the proof of Frobenius's Theorem. No reference to such credit is made in the present edition.

The most striking feature of the new edition is the emphasis on plane symmetries and on the theory of ornamental figures. These subjects are here recommended as suitable first steps towards an understanding of abstract group theory. The present volume aims to provide three distinct entrances to this subject. The first of these starts from the postulates, the second begins with ornamental figures, while the third employs the classic theory of substitutions, which is here called the theory of permutations. The wide differences between these methods of entering into the same general field tend to interest the student and to exhibit at the outset the extensive range of applications of the general theory.

The present volume aims also to arouse a deep historical interest by referring to the fundamental question of the source and the nature of the earliest higher mathematics in ancient times. The view is expressed here that this source is found in architecture and in ornamental figures, and hence the earliest study of higher mathematics was inspired by a sense of beauty rather than by an interest in elementary geometry, which is necessarily a somewhat tedious subject. It thus appears that the history of higher mathematics may begin about 1000 years earlier than has been commonly assumed by mathematical historians (p. 3). We refer here to these historical remarks, some of which will probably not be accepted

without protest, to exhibit the attractive style in which the subject matter of the present volume has been presented. The fact that the author sometimes expresses general views which may possibly be challenged may tend to awaken interest and discussions on the part of the readers.

The number of chapters has been increased from 15 to 16 while the number of sections has been extended from 63 to 74. An "Einleitung" has been added, in which prehistoric group theory and the derivation of the group concept from permutations is considered. The heading of the last section has been changed from "Anwendung der Substitutionsgruppen" to "Die Kleinsche Gleichungstheorie," but besides the correction of slight errors and the addition of a foot-note no changes were made in this section. In the list of authors the name of *Owen Jones* appears under the letter *O* instead of under the letter *J*. The developments relating to abelian groups have been greatly extended but the $\phi$-subgroups did not receive any attention. As in the first edition so in the present one the applications are expecially stressed in accord with the expressed purpose of the valuable series of which the present volume is a worthy part.

G. A. MILLER

*Das Rechnen der Naturvölker*. By E. Fettweis. Leipzig, B. G. Teubner, 1927. iv+96 pp. Price 5 marks.

This is a compilation of the number words used by many savage races. Gestures for numbers and some of the extremely simple calculations which these races make, are also described. An extensive bibliography of nearly 300 titles is given. It does not include L. L. Conant's *Number Concept*, New York, 1896, though the author makes a second-hand allusion to Conant, (p. 54); and it has few titles in common with the bibliography prepared from the mathematical standpoint by David Eugene Smith, (*History of Mathematics*, vol. 1, p. 14). The work seems likely to be of more interest to the ethnologist, the linguist and, perhaps, the psychologist, than to the mathematician.

J. W. A. YOUNG

*Ptolémée, Composition Mathématique, traduite pour la première fois du grec en français par M. Halma (avec le texte grec): et suivie des notes de M. Delambre*. Paris, J. Hermann, Volumes 1, 2. Réimpression facsimilé. Price 210 fr.

The edition of Ptolemy's *Almagest*, in Greek and French, printed in parallel columns, which was prepared in 1813–16 by the noted French scholar l'Abbé Halma and supplied with notes by the astronomer Delambre, is generally regarded as the best in existence. Before that time a translation from Greek into Latin had appeared at Basel which was based on a less careful scrutiny of different manuscript texts. Halma made a searching comparison of manuscripts in the libraries in Paris, Venice, Florence and the Vatican. Now, more than a century after the first appearance of Halma's edition, a facsimile reimpression is placed within the reach of readers of our time. It is well that Ptolemy should be available to

modern students in a modern language, for Ptolemy's Almagest is the text from which astronomers have drawn knowledge and inspiration for over a thousand years, a longer period of authority than that of any other astronomical author.

The first volume of Halma's edition begins with an elaborate Preface in French which is an historical and critical account of Ptolemy's great book, emphasizing its value to the modern astronomer. This part, though rich in detail, might have been profitably rewritten so as to embody later studies due to Paul Tannery and others. An historical result so very recent that it could not have been included even in a revision, unless written since 1926, is the demonstration that the precession of the equinoxes was known before the time of Hipparchus to the Babylonian astronomer Kidinnu.

Of special historical interest to mathematicians is Ptolemy's spherical trigonometry which he develops more fully than plane trigonometry. In his computation of a table of "chords" he establishes the theorem on the inscribed quadrilateral, that the product of its diagonals is equal to the sum of the products of the opposite sides. A simple and elegant geometric construction (*Ptolemy* I, p. 27), which yields simultaneously the sides of a regular inscribed pentagon and decagon is known to modern engineers, but is not given in elementary texts on geometry because of the comparatively greater difficulty of the proof.

FLORIAN CAJORI

*Sieben- und mehrstellige Tafeln der Kreis- und Hyperbelfunktionen und deren Produkte sowie der Gammafunktion, nebst einem Anhang: Interpolations- und sonstige Formeln.* By Keiichi Hayashi. Berlin, Julius Springer, 1926. vi+283 pp.

A review of these tables by T. H. Gronwall has already appeared in this Bulletin (vol. 32 (1926), p. 718). Dr. Gronwall's review concerned the contents and scope of the tables; the present review relates to their accuracy.

Wishing to construct a table of two functions auxiliary to the hyperbolic functions, and having read Dr. Gronwall's review, I decided that these tables would facilitate the compilation. Accordingly I procured a volume and used it. A computation formula was used which checked not only the computed values but also the consistency of the tabular values from which they were obtained, and the results have been rather disconcerting.

The first error encountered was in $\sinh x$, $x=.0783$ to $x=.0799$, in which a column of 7's should be a column of 8's. This however is quite probably a proof-reader's error and such errors may easily be corrected by the publisher. But in $\sinh .548$ and $\cosh .548$ the tabulated values are each too large by 0.000005 and the chances of this being an error in proof-reading are small. The chances are diminished by the fact that the same kind of thing occurs in eight other places, and for $x=.782$ and .923 the corresponding error is carried into $e^x$; $\cosh .872$ and $\sinh .872$ are each in error by 0.0000091618! Such errors could hardly arise from bad proof-reading and one wonders what method of systematic checking would fail to reveal them.

All told, from the calculations made which would involve roughly 5% of the digits of the table, 25 groups of errors, as shown below, have been discovered which involve 55 functional values and 81 digits. Further argument seems unnecessary. One cannot fail to have the deepest sympathy with Mr. Hayashi in the enormous loss and disappointment which he sustained when his first calculations were as he states destroyed by fire, and to have the greatest admiration for his perseverance in immediately duplicating the entire work. But after all the value of the table is a matter of its accuracy and until the whole is thoroughly checked the table must be regarded as unreliable and used with caution. When made dependable these tables will be a great asset to any library.

ERRORS IN HAYASHI'S TABLES

| *Function* | | *Tabular Value* | *Correct Value* | *Error (T–C)* |
|---|---|---|---|---|
| sinh | .0783 | 0.0783700326 | 0.0783800326 | −0.00001 |
| " | .0784 | 4703397 | 4803397 | " |
| " | .0785 | 5706476 | 5806476 | " |
| " | .0786 | 6709563 | 6809563 | " |
| " | .0787 | 7712657 | 7812657 | " |
| " | .0788 | 8715760 | 8815760 | " |
| " | .0789 | 9718870 | 9818870 | " |
| " | .0790 | 0.0790721988 | 0.0790821988 | " |
| " | .0791 | 1725114 | 1825114 | " |
| " | .0792 | 2728248 | 2828248 | " |
| " | .0793 | 3731390 | 3831390 | " |
| " | .0794 | 4734540 | 4834540 | " |
| " | .0795 | 5737698 | 5837698 | " |
| " | .0796 | 6740864 | 6840864 | " |
| " | .0797 | 7744037 | 7844037 | " |
| " | .0798 | 8747219 | 8847219 | " |
| " | .0799 | 9750409 | 9850409 | " |
| tanh | .199 | 0.19461409 | 0.19641409 | −0.0018 |
| " | .332 | 0.22034672 | 0.22031672 | −0.00003 |
| sinh | .548 | 0.5758475557 | 0.5758425557 | +0.000005 |
| cosh | " | 1.1539524204 | 1.1539474204 | " |
| sinh | .554 | 0.5827816469 | 0.5827766469 | +0.000005 |
| cosh | " | 1.1574282675 | 1.1574232675 | " |
| sinh | .672 | 0.7237367615 | 0.7237317615 | +0.000005 |
| cosh | " | 1.2344229449 | 1.2344179449 | " |
| sinh | .719 | 0.7825752763 | 0.7825702763 | +0.000005 |
| cosh | " | 1.2698145280 | 1.2698095280 | " |

| *Function* | | *Tabular Value* | *Correct Value* | *Error (T–C)* |
|---|---|---|---|---|
| sinh | .745 | 0.8158385675 | 0.8158535675 | −0.000015 |
| cosh | " | 1.2905728675 | 1.2905878675 | " |
| $e^{.782}$ | | 2.1858385759 | 2.1858395759 | −0.000001 |
| sinh | .782 | 0.8641742302 | 0.8641747302 | −0.0000005 |
| cosh | " | 1.3216643457 | 1.3216648457 | " |
| cosh | .838 | 1.3721586405 | 1.3721568405 | +0.0000018 |
| sinh | .872 | 0.9867783226 | 0.9867874844 | −0.0000091618 |
| cosh | " | 1.4049111297 | 1.4049019679 | + " |
| $e^{.923}$ | | 2.5168265642 | 2.5168295642 | −0.000003 |
| sinh | .923 | 1.0597506444 | 1.0597521444 | −0.0000015 |
| cosh | " | 1.4570759199 | 1.4570774199 | " |
| sinh | .928 | 1.0671508087 | 1.0670508087 | +0.0001 |
| cosh | 1.137 | 1.7189910180 | 1.7190910180 | −0.0001 |
| sinh | 1.361 | 1.8218835997 | 1.8218435997 | +0.00004 |
| cosh | 1.363 | 2.0819906903 | 2.0818956903 | +0.000005 |
| sinh | 1.615 | 2.4145986265 | 2.4144986265 | +0.0001 |
| cosh | 1.933 | 3.5374616062 | 3.5274616062 | +0.01 |
| sinh | 1.942 | 3.4164327851 | 3.4146327851 | +0.0018 |
| cosh | 1.952 | 3.5823744152 | 3.5923744152 | −0.01 |
| " | 1.953 | 858265966 | 958265966 | " |
| " | 1.954 | 892823738 | 992823738 | " |
| cosh | 2.422 | 5.6725587463 | 5.6785587463 | −0.006 |
| cosh | 2.554 | 6.4680023773 | 6.4681023773 | −0.0001 |
| $e^{2.717}$ | | 15.1348491214 | 15.1348495214 | −0.0000004 |
| sinh | 2.717 | 7.5343932230 | 7.5343884230 | +0.0000048 |
| cosh | " | 7.6004658984 | 7.6004610984 | +0.0000048 |
| sinh | 2.789 | 8.1015321270 | 8.1016321270 | −0.0001 |
| sinh | 2.803 | 8.2164134403 | 8.2167134403 | −0.0003 |

J. W. CAMPBELL

*Lehrbuch der Algebra.* By Robert Fricke. Volume II. *Ausführungen über Gleichungen niederen Grades.* Vieweg und Sohn, Braunschweig, F., 1926. vii+418 pp.

Fricke's Algebra is a worthy successor to Weber's Algebra, which it henceforth displaces. The present volume is a very attractive exposition of the modern theory of equations of degrees 5, 6, 7. The approach is quite different from that by Weber and more attractive since extensive use is made of the geometric and function-theoretic methods developed by Klein in his *Ikosaeder* and books on elliptic modular functions. Fricke's long experience with the latter subject made it easy for him to give a simple authoritative exposition of those portions of it which suffice for the transcendental solutions of equations of low degrees.

Part I treats finite groups of linear substitutions on two variables and quintic equations (181 pages). Part II deals with finite linear groups on three variables and equations of degrees 6 and 7 (158 pages). Part III gives applications to the inflexion points of a plane cubic curve and to the bitangents to a quartic curve (85 pages); it is similar to Weber's account, but with important improvements.

Any one interested either in the theory of algebraic equations or in elliptic modular functions will find this book indispensable.

L. E. Dickson

*The Theory of Functions of a Real Variable and the Theory of Fourier's Series.* By E. W. Hobson. Volume I. Third edition. Cambridge University Press, 1927. xv+736 pp.

For reviews of volumes I and II of the second edition see this Bulletin, vol. 28 (1922), pp. 266–270, and vol. 33 (1927), pp. 115–118, respectively. In preparing the third edition of volume I the author has revised his work throughout and has enlarged the volume by more than sixty pages. (He has also included three pages of additions and corrections to volume II.) There has been no change in plan, and section numbers have been left unaltered so that the references in volume II to the material in volume I are still applicable. Various sections have been rewritten and several new sections have been inserted. In the theory of Riemann-Stieltjes integration particularly changes and additions have been made. The references are ample though they do not provide a means of compiling a complete bibliography of the subject; they are rendered more useful in this new edition by the insertion of dates. In view of the size of the volume and the extraordinary richness of its material, the short general index of but little more than two pages seems quite inadequate; even after having read the volume with some care, one frequently finds it unduly difficult, owing to the inadequacy of the index, to find the treatment of a specific topic which he wishes to reexamine. The improvements and the addition of new matter in this edition will increase the value of this work, already far and away the most useful existent treatise on the theory of functions of a real variable.

R. D. Carmichael

*Étude Élémentaire de l'Hyperbole Équilatère et de quelques Courbes Dérivées.* By J. Lemaire. Paris, Vuibert, 1927. 172 pp

Book one contains a detailed exposition of the elementary properties of the equilateral hyperbola; much attention is given to the osculating circles and the bitangent circles. The second book is concerned with the strophoid and lemniscate as inverses of the equilateral hyperbola, and with the three-cusped hypocycloid. The many properties of the osculating and of the bitangent circles of these inverse curves follow from the detailed treatment in book one. The discussion of these curves is more extensive than that in any standard treatise. The development is synthetic, well ordered, and the book is easy reading. The text is supplemented by nearly one hundred exercises.

On page 163, example 58, a 3-bar link-work is defined which gives an easy mechanical construction for the lemniscate. The author, apparently following Teixeira, attributes this to Carbonnelle (Lemaire says "Carbonelle"). Carbonnelle proposed this in Nouvelle Correspondance (vol. 5 (1879), pp. 220 and 249). But the theorem had already been given by Phillips in the American Journal of Mathematics vol. 1 (1878), p. 386).

B. H. Brown

*Coup d'Oeil sur la Théorie des Déterminants Supérieurs dans son Etat Actuel.* By Maurice Lecat. Bruxelles, Lamertin. 1927, viii+97+100 pp.

This synopsis of the present state of hyperdeterminants is a continuation of the researches of M. Lecat on the properties of determinants of $N$ dimensions. It is to be followed by a treatise in three volumes, soon to appear, which will include the applications. The synopsis contains 19 chapters and a bibliography. It is intended to be of a critical character, the author remarking that almost all the investigations before 1910 in this field were incorrect, or at least spotted with serious errors. The exceptions are Cayley and Sylvester. Gegenbauer's work is cited as containing many errors, his results on the adjoint and upon skew symmetric determinants being completely wrong.

This field of course is intimately connected with matrices of cubic, and higher arrays, and the first thing taken up is the "topology" of such arrays. In the next chapter "activity" is considered. This property belongs to a matrix when the elements (written with subscripts) are symmetric as to at least two subscripts. The succeeding two chapters introduce considerable terminology. The next four chapters deal with developments of the determinant. In the second part of the synopsis, which has a separate paging, though the chapters run consecutively, is considered multiplication of determinants, and some special forms, such as continuants, adjoints, circulants, etc.

The synopsis will be useful to the student who desires to follow up the investigations of this field, particularly the very important ones of M. Lecat himself. It really demands the "*Traité*" to make it thoroughly useful.

J. B. Shaw

# NOTES

The April, 1928, number of the Transactions of this Society (volume 30, No. 2) contains the following papers: *The foundations of a theory in the calculus of variations in the large*, by M. Morse; *Topological invariants of knots and links*, by J. W. Alexander; *On the expansion of analytic functions in series of polynomials and in series of other analytic functions*, by J. L. Walsh; *Primitive groups which contain substitutions of prime order p and of degree 6p or 7p*, by M. J. Weiss; *Generalized Lagrange problems in the calculus of variations*, by C. F. Roos; *On Jacobi's arithmetical theorems concerning the simultaneous representation of numbers by two different quadratic forms*, by J. V. Uspensky; *On relative content and Green's lemma*, by H. L. Smith; *On Bell's arithmetic of Boolean algebra*, by W. A. Hurwitz; *A theorem on orthogonal functions with an application to integral inequalities*, by L. L. Dines; *A theorem on orthogonal sequences*, by L. L. Dines.

The April, 1928, number of the American Journal of Mathematics (volume 50, No. 2) contains: *A theorem on correspondences on algebraic curves*, by S. Lefschetz; *Concerning the structure of a continuous curve*, by G. T. Whyburn; *Compound statements on four classes*, by F. E. Baker; *A memoir on the invariants of biternary quantics*, by O. E. Glenn; *Transitive substitution groups containing regular subgroups of lower degree*, by F. E. Johnston; *Generalized Green's functions for compatible differential systems*, by W. W. Elliott; *An expansion involving p inseparable parameters associated with a partial differential equation*, by C. C. Camp; *On the asymptotic evaluation of functions defined by contour integrals*, by D. M. Wrinch; *Generalized quaternion algebras and the theory of numbers*, by L. W. Griffiths.

It is announced by the Chairman, Professor J. C. Fields, that the Proceedings of the Mathematical Congress held at Toronto in 1928 will be published while this issue of this Bulletin is in press. There will be two volumes aggregating nearly 2000 quarto pages, bound in cloth. Among other things, there will be found in the first volume a group photograph of the Congress, and the *procès verbal* of the meeting of the International Mathematical Union held during the Congress. In addition to the eight lectures printed in the first volume, the Proceedings will include about one hundred fifty communications printed in full, and over fifty abstracts. There are about one hundred eighty authors, representing twenty-six countries. The price of the entire Proceedings will be five dollars plus postage to registered members of the Congress, and ten dollars plus postage to others; orders may be sent to the University of Toronto Press. Separate copies of the group photograph may be had at fifty cents each.

The papers read at the bicentenary Newton celebration held in New York under the auspices of the History of Science Society (see this Bulletin, vol. 34, p. 247) have been published in book form with the title *Sir Isaac*

*Newton, 1727–1927, A Bicentenary Evaluation of his Work.* The book was prepared for publication by F. E. Brasch, and has an introduction by D. E. Smith; it is published by Williams and Wilkins, of Baltimore.

In response to a request by G. Albini, Senator, Rector of the University of Bologna, and President of the Organization Committee of the International Congress of Mathematicians, the American Mathematical Society has named the following six persons as its representatives: Professors Archibald, Birkhoff, Blichfeldt, Kasner, Snyder, Veblen. If a chairman be necessary, Professor Birkhoff will act.

The French Association for the Advancement of Science will meet at La Rochelle, July 23–28, 1928. Professor L. G. Du Pasquier, of the University of Neufchatel, is president of the section of mathematics.

The Louisiana Academy of Sciences held its first annual meeting at Louisiana College, Pineville, May 5, 1928. Professor I. Maizlish, of Centenary College, is president, and Professor H. L. Smith, of Louisiana State University, vice-president. Professor H. A. Wilson, of Rice Institute, delivered the principal address, entitled *Recent theories of atomic structure.* Mathematical papers were read by Professors J. A. Hardin, C. D. Smith, and H. L. Smith, and Mrs. Yetta V. Maizlish. The next annual meeting will be held at the Southwestern Louisiana Institute, Lafayette.

The Belgian Academy of Sciences has awarded its prize for a contribution to the resolution of problems with axial symmetry in general relativity (see this Bulletin, vol. 32, p. 402) to Maurice Nuyens, of the University of Brussels, for his memoir entitled *Champ gravifique dû à un électron animé d'un mouvement de rotation.*

A prize in honor of U. Dini will be conferred by the University of Pisa for a manuscript or printed work in pure mathematics, mathematical physics, celestial mechanics, or theoretical geodesics; only graduates or students of the University of Pisa, and graduates of other universities who have studied at the University of Pisa since their graduation, are eligible. The competition closes in December, 1928.

Competition for the first award of the Lazzario Fubini prize of the Unione Matematica Italiana (see this Bulletin, vol. 31, p. 282) closes December 31, 1928; memoirs should be sent to Professor E. Bortolotti, of the University of Bologna, before that date. This prize is to be awarded every two years and is restricted to Italian graduates of Italian universities who received their degrees within six years of the closing date.

Dr. Paul Koebe, professor of mathematics at the University of Leipzig, has been awarded the international mathematics prize of the King of Sweden, on recommendation of the Swedish Academy of Sciences, for his memoir *Allgemeine Theorie der Riemannschen Mannigfaltigkeiten (konforme Abbildung und Uniformisierung).*

Cambridge University has awarded Smith prizes to W. L. Edge, of Trinity College, for an essay on *Ruled surfaces of the fourth, fifth and sixth orders,* and to A. H. Wilson, of Emmanuel College, for an essay on *The*

*two-centre problem in wave mechanics.* Rayleigh prizes have been awarded to J. A. Gaunt, of Trinity College, for an essay on *The foundation of the Debye-Hückel ionization theory with application to gases*, and to W. H. M'Crae, of Trinity College, for an essay on *The quantum theory and the specific heat of gases.*

Dr. W. Nernst, professor of physical chemistry in the University of Berlin, has been awarded the Franklin medal of the Franklin Institute.

The Louis Edward Levy medal of the Franklin Institute has been awarded to Dr. Vannevar Bush, professor of electric power transmission at the Massachusetts Institute of Technology, for two papers on the product integraph, an instrument developed under Dr. Bush's direction.

Dr. W. H. Wright, astronomer of the Lick Observatory, has been awarded the Henry Draper medal of the National Academy of Sciences, for his researches on nebulae, new stars, and planetary atmospheres.

Professor Paul Alexandroff, of the University of Moscow, has been elected a corresponding member of the Göttingen Scientific Society.

Professor Harry Bateman, of the California Institute of Technology, has been elected a fellow of the Royal Society of London.

The Montanistische Hochschule in Leoben has conferred an honorary doctorate on Professor E. Doležal, of Vienna.

Professor G. Kowalewski has been elected a member of the Saxon Academy of Sciences.

Professor Louis Roy, of the Faculty of Sciences of Toulouse, has been elected a correspondent of the Paris Academy of Sciences in the section of mechanics.

Professor Albert Einstein has been elected an honorary member of the British Physical Society.

Dublin University has conferred an honorary doctorate on Professor A. S. Eddington, of Cambridge.

The University of Liverpool has conferred an honorary doctorate of science on Professor J. E. Littlewood, of Cambridge.

Professor J. D. Tamarkin, of Brown University, has been elected a member of the American Academy of Arts and Sciences.

Dr. K. Menger, of the University of Vienna, has been promoted to an associate professorship.

Dr. Leopold Vietoris, of the University of Vienna, has been appointed to an associate professorship at the University of Innsbruck.

At the University of Paris, the chair of the theory of groups and the calculus of variations has been changed to a chair of the theory of functions and the theory of transformations, to which Professor E. Vessiot has been appointed; Professor R. Garnier succeeds him in his former chair.

At the University of Besançon, Professor J. Haag, of the University of Clermont, has been appointed to the chair of rational and applied mechanics, as successor to Professor J. Andrade, retired.

At the University of Dijon, the chair of pure mathematics has been changed to a chair of the differential and integral calculus to which Dr. Lagrange, of the faculty of sciences of Aix-Marseille, has been appointed.

Professor Georges Cerf, of the University of Dijon, has been appointed to the professorship of general mathematics at the University of Strasbourg.

Dr. Adolph Fraenkel, professor of mathematics at the University of Marburg, has been called to the University of Kiel.

Dr. A. Hammerstein, of the University of Berlin, has been promoted to an associate professorship.

Associate Professor Friedrich Hartogs, of the University of Munich, has been promoted to a full professorship.

Professor Otto Hölder, of the University of Greifswald, has retired.

Dr. K. Reinhardt, of the University of Leipzig, has been promoted to a professorship.

Dr. G. Pólya has been appointed to a professorship at the Zurich Technical School, as successor to Professor F. Rudio, retired.

Dr. R. K. Butchart, lecturer in mathematics at the University College, Dundee (the University of St. Andrews) has been appointed professor of mathematics at Raffles College, Singapore.

The electors of the newly established Rouse Ball professorship of mathematics at the University of Cambridge have elected J. E. Littlewood, fellow of Trinity College and Cayley lecturer, to the professorship.

Professor Constantin Carathéodory of the University of Munich, Professor G. C. Evans of the Rice Institute, and Professor H. B. Phillips of the Massachussetts Institute of Technology, will lecture at the University of California during the summer of 1928.

Professor Herman Weyl, of the Zurich Technical School, has been appointed to the Thomas D. Jones research professorship of mathematical physics at Princeton University.

At Princeton University, Associate Professors J. W. Alexander, S. Lefschetz, C. R. MacInnes, and J. H. M. Wedderburn have been promoted to full professorships of mathematics.

Professor Daniel Buchanan, head of the department of mathematics at the University of British Columbia, has been appointed Dean of the Faculty of Arts and Science at that university.

At the University of Syracuse, Associate Professor Alan D.Campbell has been promoted to a professorship of mathematics.

Professor G. W. Gorrell has been made head of the department of mathematics at the University of Denver.

Assistant Professor Charles Duncan Gregory, of William and Mary College, has been promoted to an associate professorship of mathematics.

Dr. C. E. Hille of Princeton, and Dr. Alfred Errera of Belgium, will lecture at Stanford University during the summer session.

At Yale University, Professor W. R. Longley has been appointed to the Colgate professorship, as successor to Professor Luquiens; and Dr. L. T. Moore has been promoted to an assistant professorship of mathematics.

On account of blindness, Professor T. E. McKinney, of the department of mathematics at the University of South Dakota, will retire from active service at the end of the present academic year.

Dr. Max Mason, president of the University of Chicago and formerly professor of mathematical physics at the University of Wisconsin, has resigned to accept an appointment as director of the newly created division of sciences of the Rockefeller Foundation.

At Columbia University, Associate Professor G. W. Mullins, of Barnard College, has been promoted to a full professorship of mathematics; Dr. R. G. Archibald and Dr. B. O. Koopman have been promoted to assistant professorships of mathematics; and Dr. Lulu Hofmann has been appointed to a lecturership in mathematics.

Dr. C. H. Richardson, of Georgetown College, Kentucky, has been appointed head of the department of mathematics at Bucknell University, as successor to Professor W. C. Bartol, who has retired after forty-seven years of service.

Professor J. E. Rowe, of the College of William and Mary, has resigned to accept the presidency of the Clarkson Memorial Institute of Technology at Potsdam, N. Y.

At the University of California at Los Angeles, Assistant Professor H. M. Showman has been made Recorder of the University but will remain a member of the department of mathematics, and Assistant Professor Myrtie Collier has withdrawn; Dr. Raymond Garver of the University of Rochester, and Professor W. M. Whyburn, formerly of the Texas Technological College, and recently National Research Fellow, have been appointed to assistant professorships of mathematics.

Dr. D. J. Struik has been appointed assistant professor of mathematics at the Massachusetts Institute of Technology.

Assistant Professor J. D. Tamarkin, of Brown University, has been promoted to a full professorship of mathematics.

The following appointments to instructorships are announced:

Albion College, Mr. Watson M. Davis.

The University of Arkansas, Mr. Harvey A. Wright.

Columbia University, Mr. M. S. Demos.

The University of Chicago, Dr. B. W. Jones.

Dartmouth College, Dr. F. W. Perkins.

Harvard University, Dr. M. M. Slotnick (National Research Fellow), A. C. Berry, A. B. Brown, A. H. Fox, G. A. Hedlund, J. J. Hinrichsen, J. K. Peterson, G. B. Price, G. Saute, and Charles Wexler.

St. Olaf's College, Mr. Clarence Carlson.

Professor Henri Bosmans, of the Jesuit College of Saint-Michel, Brussels, died February 3, 1928, at the age of seventy-six. He was known for his work in the history of mathematics.

Professor L. Schleiermacher, of the Technical School at Darmstadt, died November 10, 1927.

Professor Emil Waelsch, of the German Technical School at Brunn, died June 5, 1927.

Professor Giuseppe Sforza died December 17, 1927.

Professor Emilio Veneroni, president of the Royal Technical School at Piacenza, and docent in geometry at the University of Padova, died June 18, 1927.

Professor W. S. Aldis, of Auckland College, New Zealand, died March 7, 1928, at the age of eighty-nine.

Mr. W. B. Croft, formerly of the department of mathematics at Winchester College, died March 23, 1928, at the age of seventy-six.

Dr. J. B. Chittenden, professor of mathematics at the Brooklyn Polytechnic Institute, died March 20, 1928, at the age of sixty-four.

Professor Gaetano Lanza, emeritus professor of theoretical and applied mechanics at the Massachusetts Institute of Technology, died March 21, 1928, at the age of seventy-nine.

Dr. F. S. Luther, emeritus president and formerly professor of mathematics at Trinity College, Hartford, died January 4, 1928, at the age of seventy-seven.

# NEW PUBLICATIONS

## PART I. PURE MATHEMATICS

ACKERMANN (W.). See HILBERT (D.).

ANDRADE (E. N. DA C.). See LUCRETIUS.

APPELL (P.). See POINCARÉ (H.).

BOREL (E.). Leçons sur les séries divergentes. 2e édition revue et entièrement remaniée avec le concours de Georges Bouligand. (Borel Series.) Paris, Gauthier-Villars, 1928. 260 pp.

—— Leçons sur la théorie des fonctions. 3e édition. Paris, Gauthier-Villars, 1927. 290 pp.

BOULIGAND (G.). See BOREL (E.).

BROGGI (U.). Análisis matemático. Volumen II: Teorías generales. Funciones de más de una variable. La Plata, Universidad Nacional de La Plata, 1927. 211 pp.

CARSLAW (H. S.). Introduction to the theory of Fourier's series and integrals. (In Japanese.) Translated by G. Takemae. Tokyo, Uchida Rokakuho, 1927. 12+482 pp.

CARTAN (E.). See DELENS (P. C.).

CASSINIS (G.). Calcolo numerici, grafici e meccanici. Pisa, Mariotti-Paccini, 1928. 20+672 pp.

CORRAL (J. I.). Nuevas formulas de transformacion de coordenadas. Habana, 1928. 91 pp.

CURTISS (D. R.) and MOULTON (E. J.). Trigonometry, plane and spherical. New York, Heath, 1927. 11+264 pp.

DELENS (P. C.). Méthodes et problèmes de géométries différentielles, euclidienne et conforme. Avec une préface de M. E. Cartan. Paris, Gauthier-Villars, 1927. 10+184 pp.

DRACH (J.). See POINCARÉ (H.).

DURELL (C.) and WRIGHT (R. M.). Elementary trigonometry. London, Bell, 1927. 18+288+31+24 pp.

FLADT (K.). Euklid. Berlin, Salle, 1927. 72 pp.

FRANZ (W. E.). $X^n+Y^n \gtrless Z^n$. Berlin, Verein für Soziale Ethik und Kunstpflege, 1927.

GOURSAT (E.). Cours d'analyse mathématique. 4e édition. Tome III. Paris Gauthier-Villars, 1927. 702 pp.

GÜNTHER (S.). Geschichte der Mathematik. Teil 1: Von den ältesten Zeiten bis Cartesius. Berlin, de Gruyter, 1927. 7+429 pp.

HEFFTER (L.) und KOEHLER (C.). Lehrbuch der analytischen Geometrie. 2te wesentlich umgearbeitete und vermehrte Auflage. Band 1. Karlsruhe, G. Braun, 1927. 17+477 pp.

HILBERT (D.) und ACKERMANN (W.). Grundzüge der theoretischen Logik. (Die Grundlehren der Mathematischen Wissenschaften, Band 27.) Berlin, Springer, 1928. 8+120 pp.

HONECKER (M.). Gegenstandslogik und Denklogik. 2te Auflage. Berlin, Dümmler, 1928.

KLEIN (F.). Elementarmathematik vom höheren Standpunkte aus. Band III. Berlin, Springer, 1928. 10+238 pp.

—— Matematica elemental desde un punto de vista superior. Madrid, 1927. 369 pp.

—— Vorlesungen über nicht-euklidische Geometrie. Für den Druck neu bearbeitet von W. Rosemann. (Die Grundlehren der Mathematischen Wissenschaften, Band 26.) Berlin, Springer, 1928. 12+326 pp.

KOEHLER (C.). See HEFFTER (L.).

KOWALEWSKI (G.). Grundzüge der Differential- und Integralrechnung. 4te, verbesserte Auflage vermehrt durch einen Anhang über Fredholmsche Determinanten und Integralgleichungen. Leipzig, Teubner, 1928. 417 pp.

LAINÉ (E.). Précis d'analyse mathématique à l'usage des candidats au certificat de calcul différentiel et intégral. Tomes 1 et 2. Paris, Vuibert, 1927. 8+232+315 pp.

LANDAU (E.). Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale. 2te Auflage. Leipzig, Teubner, 1927. 7+147 pp.

LENNES (N. J.) and MERRILL (A. S.). Plane trigonometry. Five-place tables. New York, Harper, 1928. 12+178+92 pp.

LIETZMANN (W.). Aufbau und Grundlage der Mathematik. Leipzig, Teubner, 1927.

LOVE (C. E.). Analytic geometry. Revised edition. New York, Macmillan, 1927. 14+257 pp.

LUCRETIUS. T. Lucreti Cari de rerum natura libri sex. With notes and a translation by H. A. J. Munro. 4th edition, finally revised. Volume 2: Explanatory notes, with an introductory essay on the scientific significance of Lucretius by E. N. da C. Andrade. London, Bell, 1928. 22+424 pp.

MCGIFFERT (J.). Plane and solid analytic geometry. Boston, Ginn, 1928. 14+338 pp.

MACK (H.). C. F. Gauss und die seinen. Braunschweig, Appelhans, 1927. 11+130 pp.

MERRILL (A. S.). See LENNES (N. J.).

MILNE (W. P.) and WESTCOTT (G. J. B.). The elements of the calculus. London, Bell, 1927. 8+92 pp.

MOULTON (E. J.). See CURTISS (D. R.).

MULLHALL (J.). The theory of repetition. Buenos Aires, Coni, 1927. 38 pp.

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OGLESBY (E. J.). Heath logarithmic and trigonometric tables. Boston, Heath, 1927. 95 pp.

PICARD (E.). Traité d'analyse. 3e édition. Tome 3. Paris, Gauthier-Villars, 1928.

POINCARÉ (H.). Oeuvres de Henri Poincaré publiées sous les auspices de l'Académie des Sciences par Paul Appell. Tome 1, publié avec le collaboration de Jules Drach. Paris, Gauthier-Villars, 1928.

ROSEMANN (W.). See KLEIN (F.).

ROTHE (R.). See WEIERSTRASS (K.).

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SERRET (J. A.). Cours d'algèbre supérieure. 7e édition. Paris, Gauthier-Villars, 1927. 2 volumes.

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WESTCOTT (G. J. B.). See MILNE (W. P.).

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## PART II. APPLIED MATHEMATICS

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TYCHO BRAHE. Tychonis Brahe Dani opera omnia. Tomus 14. Edidit I. L. E. Dreyer. Hauniae, Libraria Gyldendeliana, 1928. 4+327 pp.

DE BROGLIE (L.). La mécanique ondulatoire. (Mémorial des Sciences Physiques, No. 1.) Paris, Gauthier-Villars, 1928. 3+54 pp.

CHARLIER (C. L.). Die Mechanik des Himmels. 2te, durchgesehene Auflage. 1ter und 2ter Band. Berlin, de Gruyter, 1927. 8+488+8+478 pp.

CHEVRIER (J.). See BELLUZZO (G.).

CHWOLSON (O. D.). Traité de physique. Tome supplémentaire: La physique de 1914 à 1926. 2e partie. Traduit du russe par M. A. Corvisy. Paris, Hermann, 1928. 289 pp.

CORVISY (A.). See CHWOLSON (O. D.).

COURTINES (M.). Où en est la physique? Paris, Gauthier-Villars, 1927. 312 pp.

CURRY (W. A.). See MORECROFT (J. H.).

DANIELS (F.). Mathematical preparation for physical chemistry. New York, McGraw-Hill, 1928. 10+308 pp.

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EBERHARD (G.), KOHLSCHÜTTER (A.), und LUDENDORFF (H.). Handbuch der Astrophysik. Band 6: Das Sternsystem. 2ter Teil. Berlin, Springer, 1928. 9+474 pp.

EDDINGTON (A. S.). Der innere Aufbau der Sterne. Nach Ergänzung der englischen Ausgabe durch A. S. Eddington ins Deutsche übertragen von E. von der Pahlen. Berlin, Springer, 1928. 8+514 pp.

FINDLAY (A.). The phase rule and its applications. 6th edition, revised and largely rewritten. London, Longmans, 1927. 15+326 pp.

FÖPPL (A.). Vorlesungen über technische Mechanik. 10te Auflage, bearbeitet von Otto Föppl. Band 3: Festigkeitslehre. Leipzig, Teubner, 1927. 4+451 pp.

FÖPPL (O.). See FÖPPL (A.).

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GEIGER (H.) und SCHEEL (K.), herausgegeben von. Handbuch der Physik. Band 12: Theorien der Elektrizität, Elektrostatik. Redigiert von W. Westphal. Band 13: Elektrizitätsbewegung in festen und flüssigen Körpern. Redigiert von W. Westphal. Band 24: Negative und positive Strahlen, zusammenhängende Materie. Redigiert von H. Geiger. Berlin, Springer, 1927, 1928, 1929. 7+564+7+672+11+604 pp.

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LAFROGNE (—.). Calcul du l'avantage du banquier au jeu de baccara. Paris, Gauthier-Villars, 1927. 42 pp.

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LÉVY (P.). Cours de mécanique professé à l'Ecole supérieure des Mines. Paris, Gauthier-Villars, 1928. 8+305 pp.

LODGE (O.). A century's progress in physics. London, University of London Press, 1927. 36 pp.

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MANN (L.). Theorie der Rahmenwerke auf neuer Grundlage. Berlin, Springer, 1927. 6+123 pp.

MARSHALL (C.R.) and GRIFFITH (H. D.). An introduction to the theory and use of the microscope. London, Routledge, 1928. 8+90 pp.

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MICHELL (A. G. M.). See PETROW (N.).

MORECROFT (J. H.), PINTO (A.), and CURRY (W. A.). Principles of radio communication. 2d edition, thoroughly revised. New York, Wiley, 1927. 14+1001 pp.

VON DER PAHLEN (E.). See EDDINGTON (A. S.).

PETROW (N.), REYNOLDS (O.), SOMMERFELD (A.), und MICHELL (A. G. M.). Abhandlungen über die hydrodynamische Theorie der Schiermittelreibung. Herausgegeben von L. Hopf. (Ostwalds Klassiker der Exakten Wissenschaften.) Leipzig, Akademische Verlagsgesellschaft, 1927.

PIETSCH (E.). See HINSHELWOOD (C. N.).

PINTO (A.). See MORECROFT (J. H.).

PLANCK (M.). Einführung in die theoretische Physik. 4te Auflage. Band 1: Einführung in die allegemeine Mechanik. Leipzig, Hirzel, 1928. 8+226 pp.

—— Vorlesungen über Thermodynamik. 8te Auflage. Berlin, de Gruyter, 1927. 10+287 pp.

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SCHRÖDINGER (E.). Abhandlungen zur Wellenmechanik. 2te Auflage. Leipzig, Barth, 1928. 10+198 pp.

SENSAUD DE LAVAUD (D.). Les vitesses critiques d'une voiture automobile. Dandinement et shimmy. Paris, chez l'auteur, 1927.

SIDGWICK (N. V.). The electronic theory of valency. Oxford, Clarendon Press, 1927. 12+310 pp.

SOMMERFELD (A.). See PETROW (N.).

SPOFFORD (C. M.). Theory of structures. 3d edition, revised and enlarged. New York, McGraw-Hill, 1928. 587 pp.

STARK (J.). Die Axialität der Lichtemission und Atomstruktur. Berlin, Seydel, 1927. 12+124 pp.

TIMOSHENKO (S.) and LESSELS (J. M.). Applied elasticity. London, Constable, 1928. 21+544 pp.

TOUSSAINT (A.). L'aviation actuelle. Etude aérodynamique et essais des avions; l'aviation actuelle et la sécurité. Paris, Alcan, 1928. 6+315 pp.

UHLENBECK (G. E.). Over statistische methoden in de theorie der quanta. (Dissertation, Leiden.) The Hague, Nijhoff, 1927. 97 pp.

WEIR (J. W.). Comets and the sun. New theories regarding their structure. New York, Longmans, 1927. 16+72 pp.

WESTPHAL (W.). See GEIGER (H.).

WILCKE (G.). See HINSHELWOOD (C. N.).

WINKEL (H.). Festigkeitslehre für Ingenieure. Nach dem Tode des Verfassers bearbeitet von K. Lachmann. Berlin, Springer, 1927.

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Meetings of the Society have been fixed at the following times and places:

AMHERST COLLEGE, SUMMER MEETING AND COLLOQUIUM, September 4-7, 1928.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than August 3. A series of colloquium lectures will be delivered by Professor A. B. Coble on *The determination of the tritangent planes of the space sextic of genus four.*

NEW YORK CITY, October 27, 1928.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than October 6.

BERKELEY, California, October 27, 1928. (The San Francisco Section.)

Abstracts must be in the hands of the Secretary of the Section, Professor B. A. Bernstein, Berkeley, California, not later than October 6. Professor E. T. Bell will give an address on *The theory of algebraic numbers in the light of Kronecker's program.*

LAWRENCE, KANSAS, December 1, 1928. (The Southwestern Section.)

Abstracts must be in the hands of the Secretary of the Section, Professor E. B. Stouffer, Lawrence, Kansas, not later than November 10.

NEW YORK CITY, ANNUAL MEETING, December 27-28, 1928.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th Street, New York City, not later than November 29.

CHICAGO, December 31, 1928-January 1, 1929.

Abstracts must be in the hands of Professor M. H. Ingraham, Associate Secretary of the Society, University of Wisconsin, Madison, Wis., not later than December 3.

R. G. D. RICHARDSON,
*Secretary of the Society.*

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Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

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